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REPORT 7666

THEORY OF RADIATION from ARRAYS AND APERTURES

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THEORY OF RADIATION FROM ARRAYS AND APERTURES

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THEORY OF RADIATION FROM ARRAYS AND APERTURES

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ABSTRACT

This report presents a physical and mathematical theory of radiation directivity * which permits the synthesis of antenna arrays to produce radiation patterns which may have arbitrarily selected intensities for all directions of radiation.

The method is equally powerful as an analysis tool.

Part I presents a physical explanation of the theory.

Part II summarizes the formulas which are used (the derivations of the formulas are presented in Appendices A and B).

Part III discusses the synthesis problem in detail.

Part IV summarizes the material presented in this report.

* [illegible]

PART I. THE GENERALIZED SPECTRUM THEORY

INTRODUCTION

This report describes and explains the application of a theoretical approach which relates the radiation-directivity pattern of an array of identical parallel antennas to the relative amplitudes and phases of the radio-frequency voltages or currents fed to the component antennas of the array. This theory is also applicable to sources of radiation wherein the energy is continuously distributed over the radiating aperture, rather than being concentrated at discrete points.

The theory is applicable to problems of analysis and to problems of synthesis, and permits a classification of arrays in terms of those characteristics which are the fundamental limitations of each class of arrays. When only a finite number of discrete radiators or collectors are available the optimum design is clearly defined subject to any set of specified conditions.

The theory shows the relations between the illumination of the aperture and the distribution of radiation through an infinite sphere concentric with the radiating aperture in free space. The aperture is the region in space occupied by the radiating elements. In this report the results are presented in terms of field strength. Power may be calculated by the usual methods if desired. The illumination is the distribution of the amplitudes and phases of the currents or voltages over the aperture, relative to a common source in transmitting arrays, and relative to the receiver input terminals in receiving arrays.

The directivity pattern of a receiving array gives the relationship between the output voltage of the array and the direction of arrival of a plane wave having a specified field strength and polarization.

The theoretical patterns are the same for receiving and transmitting arrays. The types of patterns which can be obtained with an array are determined by the shape of the aperture, the number of radiating elements (or the allowable complexity of the illumination) and the spacing between elements. The particular pattern for a given aperture is determined by the illumination.

Apertures may be divided into three broad groups.

(1) Line Sources. These include straight line or linear arrays, circular line arrays and other shaped perimeter arrays. In the theoretical limit a line source can be synthesized to approximate, arbitrarily closely, a radiation pattern defined for all azimuths at constant elevation; all elevations at constant azimuth, or, in general along such directions as can be defined by a curve drawn on a sphere which is calibrated in azimuth and elevation.

(2) Surface Sources. These include planar and non-planar area arrays; planar arrays include circular area arrays and rectangular (asparagus patch) arrays. In the theoretical limit a surface source can be synthesized to approximate, arbitrarily closely, any pattern (subject to certain symmetry restrictions) as a function of azimuth and elevation.

(3) Volume Sources. Arrays enclosed within a volume may be used to obtain any arbitrary pattern over the radiation sphere; this may often be accomplished with a smaller maximum aperture dimension than is required for surface arrays; in the case of discrete arrays it permits a large number of antennas to occupy a comparatively small region and yet be spaced far enough apart from each other so that mutual reactions are small.

THE PHYSICAL BASIS FOR DIRECTIVITY

The directivity of an array, like the selectivity of a resonant circuit, is a phase shift phenomenon. Consider two antennas such as are shown in Figure 1a. A plane wave coming from a direction defined by an azimuth angle α ,

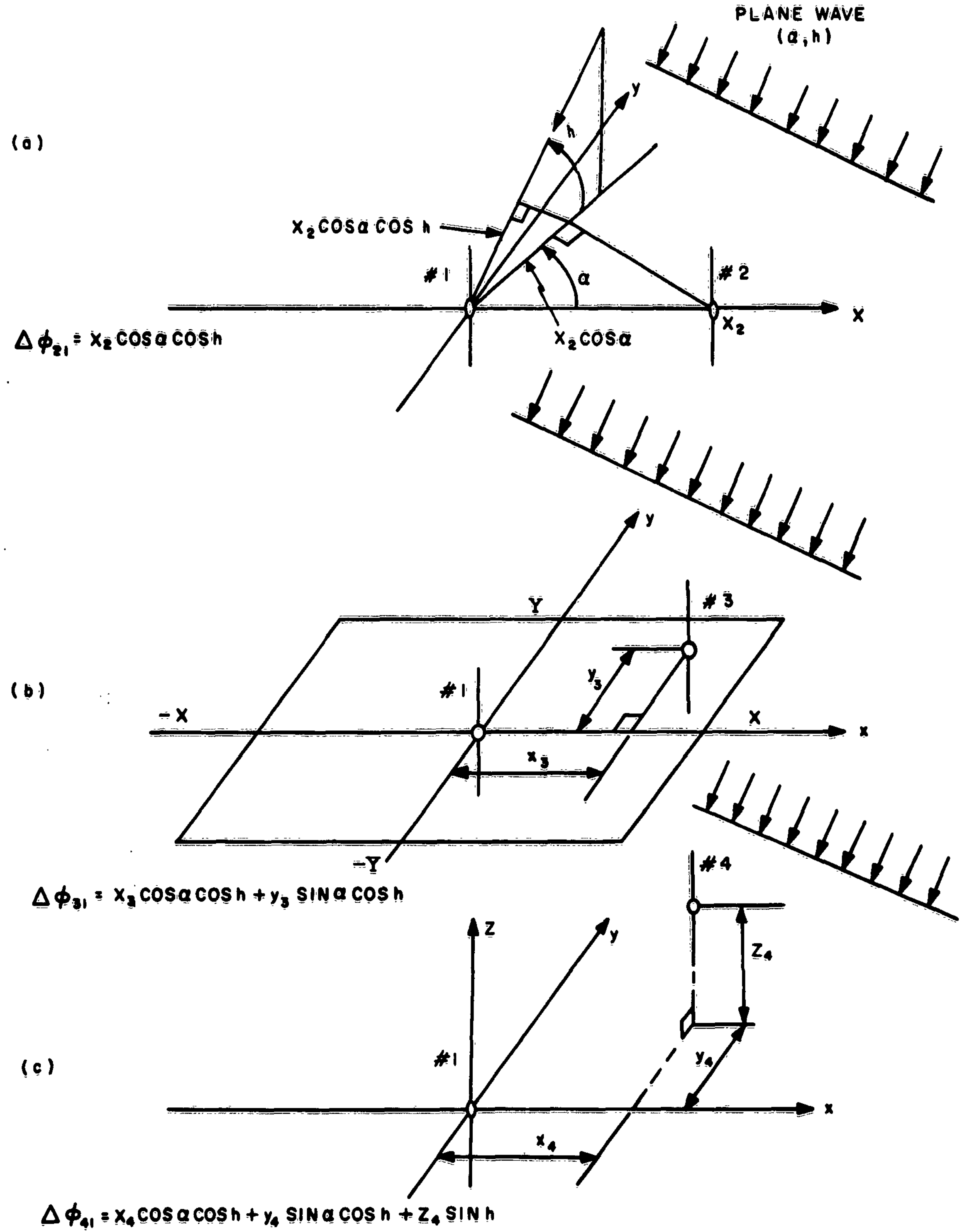

PLANE WAVE
(α, h)
(a)
h
y
$X_2 \cos\alpha \cos h$
#1
#2
α
X
X_2
$X_2 \cos\alpha$
$\Delta\phi_{21} = X_2 \cos\alpha \cos h$
(b)
y
Y
#3
#1
y_3
−X
X
x
x_3
−Y
$\Delta\phi_{31} = X_3 \cos\alpha \cos h + y_3 \sin\alpha \cos h$
(c)
Z
y
#4
Z_4
#1
y_4
x
x_4
$\Delta\phi_{41} = X_4 \cos\alpha \cos h + y_4 \sin\alpha \cos h + Z_4 \sin h$

FIG. 1

and an elevation angle h, as shown, will induce a voltage in antenna #1 at a later time than it will induce the same voltage in antenna #2. The radio frequency voltages induced may have their time-varying phases represented in the usual manner in the exponent of $\epsilon^{j\omega t}$. The space phase shift is proportional to the separation, x, between the antennas, and when the separation is measured in electrical radians the phase shift is equal to the projection of the spacing distance in the direction of the wave path. Thus, if the voltage induced in antenna 1 is $\epsilon^{j\omega(t+t_o)}$ the voltage induced in antenna 2 is

$$\epsilon^{j\left[\omega(t+t_o) + x_2 \cos h \cos \alpha\right]} \qquad (1)$$

The time-variable term is common to all the collectors of an array and will be omitted hereafter.

When there are a number of antennas spaced within a region the voltages induced in the antennas may be brought to some common point through transmission paths of known relative delay and attenuation and then added. The resultant voltage is found from the sum of these vectors. The directive pattern, or the amplitude and phase of the resultant voltage as a function of α and h, depends on how the relative phases of the several induced voltages vary with α and h; this is much like the selectivity of a tuned circuit in which the gain characteristic is determined by the phase angle between energy stored at the resonant frequency and the energy supplied at the input frequency. Because of this similarity it is reasonable that spectrum or Fourier transform analysis, which has proved such a powerful tool in circuit theory, should find its application and extension in radiation theory.

However, before exploring the meaning, interpretation and use of the spectra related to antenna arrays it is useful to introduce several useful concepts such as the geometrical space factor and to consider also how radiation patterns may be represented or thought of, and to interpret the physical reasons for the limitations of the three classes of arrays described earlier. The concepts developed here will help lay the groundwork for later considerations.

THE GEOMETRICAL SPACE FACTOR

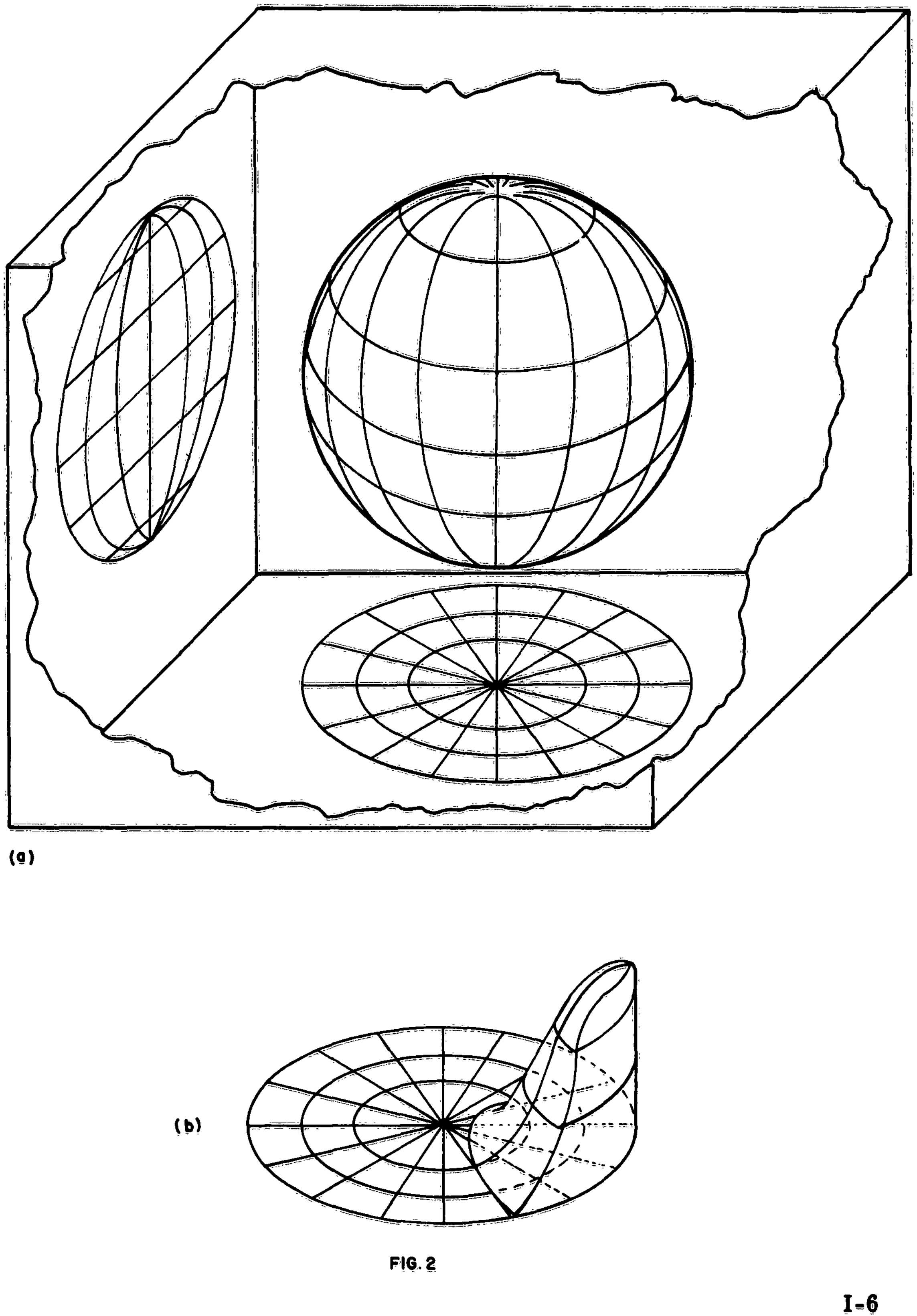
It has been customary for some time, when describing an array of identical, parallel antennas, each having a directive pattern $F_o(\alpha, h)$, to write the induced voltage of the Kth antenna as $F_o(\alpha, h)\ \epsilon^{j\phi_k}$, where ϕ_k represents the space phase shift. The factor $F_o(\alpha, h)$ is common to all the elements of the array and may be factored out when the contribution of all the antennas of the array are summed. The radiation-directivity pattern of the array is then

$$F(\alpha, h) = F_o(\alpha, h) \cdot S(\alpha, h) \qquad (2)$$

The S function is called the geometrical space factor and it is this space factor which will be dealt with hereafter. The space factor is the radiation-directivity pattern of an array of hypothetical isotropic radiators.

THE REPRESENTATION OF DIRECTIVE PATTERNS

It is convenient to represent the directive patterns of antenna arrays by their space factors relative to a sphere such as is shown in Figure 2a. Each point on the surface of the sphere, which is called the radiation sphere, has a direction from the center specified by the coordinates of azimuth and elevation; it is convenient therefore to use the symbols α and h. This has a useful physical basis not only for receiving arrays such as were described above but also for transmitting arrays for which the directive patterns are described in terms of the radiation through a sphere very large with respect to, and concentric with, the array. The patterns obtainable for receiving and transmitting antennas have the same basic physical limitations.



(a)

(b)

FIG. 2

The sphere shown in Figure 2a may have any convenient radius. It is possible to represent the space factor corresponding to any direction (α , h) by erecting an ordinate perpendicular to the surface of the sphere at each point, the ordinate being proportional to the space factor in the direction defined by that point. (While relative time delay in various directions, which is represented by the phase of the radiation in these directions, is not usually of interest, phases may be represented by a second set of ordinates on the sphere or by other means.) Alternatively, contours for equal space factor may be drawn on the surface of the sphere. In some cases, and particularly with plane arrays, it is convenient to divide the sphere into two hemispheres and to project the coordinates of each hemisphere on one or more of the planes shown in Figure 2a.

A projection on the horizontal plane which is particularly useful for arrays of antennas whose centers lie in a horizontal plane is shown in Figure 2b. Figure 2b shows a pattern of the type having a single large pencil lobe and virtually no side lobes. The amplitude under the surface represents field strength in the corresponding direction. It will be shown later how arrays may be designed to produce lobes of this type.

The patterns of space factor for an array of antennas whose centers are in a vertical plane are conveniently represented in terms of the projection of the coordinates of the radiation sphere on a vertical plane.

In general, the space factor for any array may be represented in terms of the projections, on two planes, of the coordinates of the radiation sphere.

THE LIMITATIONS OF THE THREE CLASSES OF ARRAYS

A line which represents the locus of centers of an array of isotropic antennas is shown in Figure 3a; and Figure 3b shows a curved line on the radiation sphere along which a particular desired pattern may be obtained with this class of array. If the curve of Figure 3a is divided into N separate segments representing N antennas then the curve of Figure 3b can have independently chosen space factors at N points. In the limit, therefore, an array along a line curve can be used to determine any arbitrary pattern along a contour defined by a line curve in the radiation sphere.

A surface which represents the locus of centers of an array of isotropic antennas is shown in Figure 3c. This surface is divided into N^2 sections. Figure 3d shows how the surface of the entire radiation sphere may be divided into N^2 sections, the space factor for each section being independent of that in the other sections. In the limit, therefore, a surface array can produce any desired pattern in all directions (subject to symmetry restrictions).

Figure 3e shows a volume filled with radiators. The volume may be divided into N^3 increments. Figure 3f shows the surface of the radiation sphere now divided into a number M^2 of independent increments, where $M^2 = N^3$. A volume array can produce any desired pattern, and sometimes a volume array permits a saving in array space, or in the required number of antennas. An obvious question at this point is why the volume array is not related to a volume rather than a surface; this will be answered by the material presented herein.

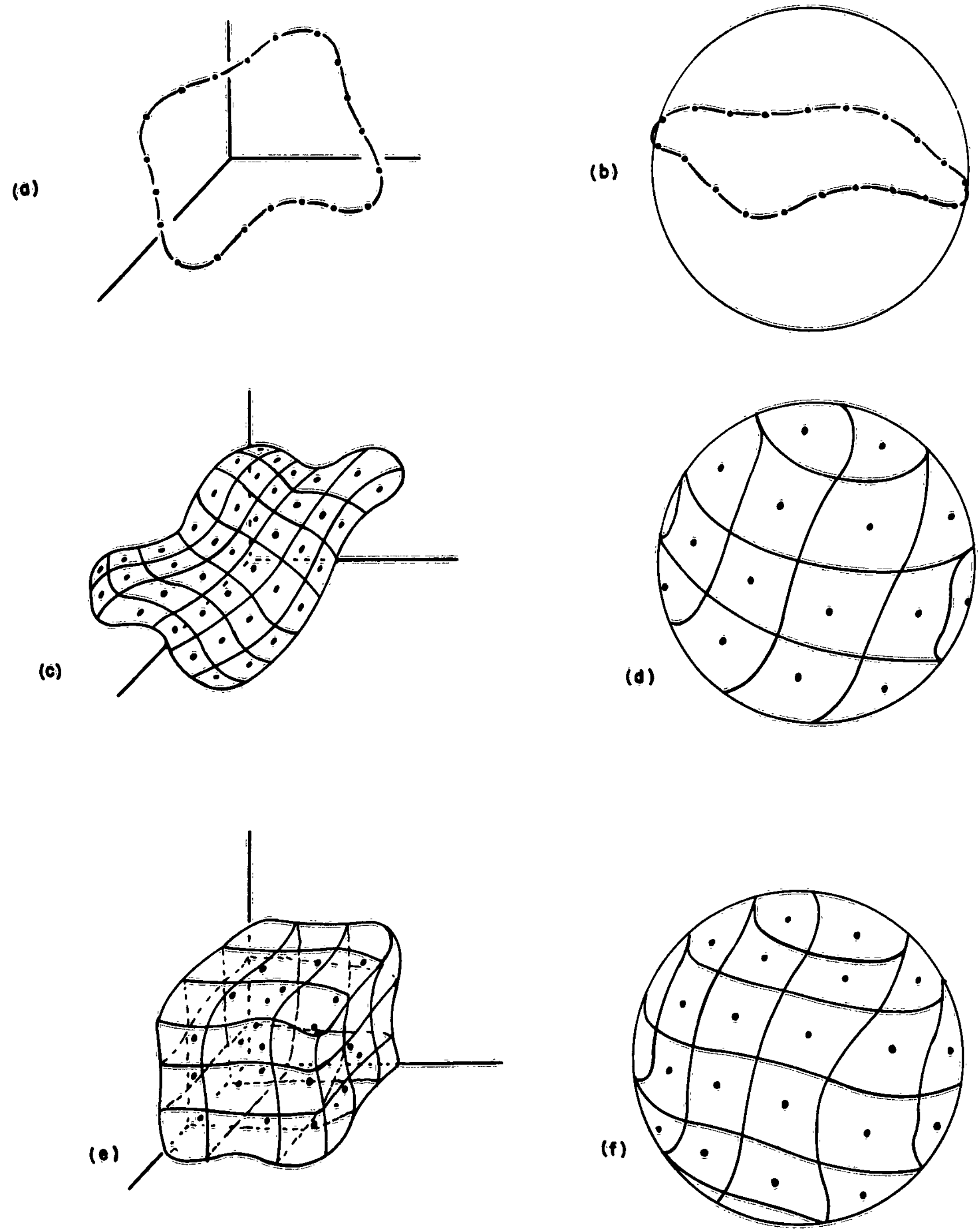


FIG. 3

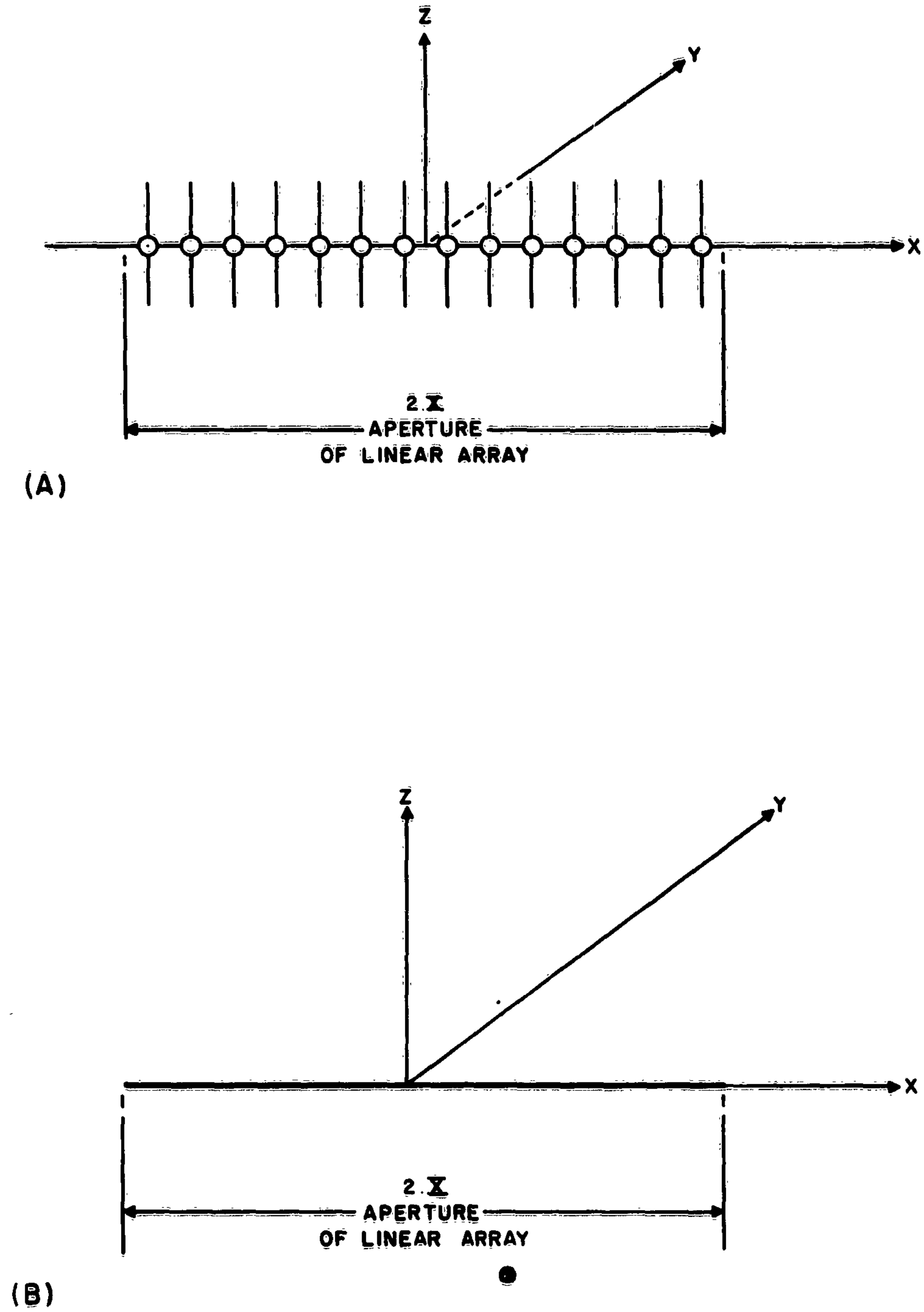
FOURIER ANALYSIS OF APERTURE ILLUMINATIONS

Some of the physical principles involved in the application of Fourier analysis and transform theory to antenna arrays can be most easily shown by first considering arrays of antennas along a straight line; such an array is shown in Figure 4a. The antennas in Figure 4a are placed along the x axis within the region (-X, X). While the grouping of antennas is arbitrary these arrays may often consist of regularly spaced antennas. The region, of length 2X, which is called the aperture, is slightly larger than the distance between the outermost antennas. This allows an equal space for each antenna and simplifies the mathematical results.

For theoretical purposes it is extremely useful to consider the antennas as continuously filling up the available region as shown in Figure 4b. This does not place any limitations upon the theory as it will be shown later how the effects of discreteness of the antennas may be considered. However, it does produce a substantial simplification of the theory since the space factor, which consists of the sum of a number of vectors, can be replaced by an integral in the case of continuous aperture illumination. This procedure yields solutions in compact, closed forms and is used for all the other types of arrays considered here as well.

The aperture illumination is represented by the function

$$I(x) = |I(x)| \epsilon^{j\phi(x)} \tag{3}$$



Z
Y
X
2 X
APERTURE
OF LINEAR ARRAY
(A)
Z
Y
X
2 X
APERTURE
OF LINEAR ARRAY
(B)

FIG.4

In the case of continuous illumination the space factor is determined by integrating, rather than adding, the effects of all the currents along the line of the array, as is shown in equation 4.

$$S(\alpha, h) = \frac{1}{2X} \int_{-X}^{X} I(x)\, \epsilon^{jx \cos\alpha \cos h}\, dx \qquad (4)$$

The types of radiation patterns which are most generally of interest are those having a pronounced major lobe. It is then convenient to introduce an initial phase adjustment such that for one direction (α_0, h_0) all of the vectors add in phase for real and positive values of I(x). A phase angle of zero in the selected direction is most convenient and therefore equation 4 can be rewritten as shown in equation 5.

$$S(\alpha, h) = \frac{1}{2X} \int_{-X}^{X} I(x)\, \epsilon^{jx(\cos\alpha \cos h - \cos\alpha_0 \cos h_0)}\, dx \qquad (5)$$

The form of equation 5 may be simplified by defining a parameter u such that

$$u = \cos\alpha \cos h - \cos\alpha_0 \cos h_0 \qquad (6)$$

The radiation pattern is then found from

$$S(u) = \frac{1}{2X} \int_{-X}^{X} I(x)\, \epsilon^{jux}\, dx \qquad (7)$$

Equation 7 has the same form as the Fourier transforms used in ordinary spectrum analysis.* I(x) is similar to the time function defining a pulse which exists entirely within specified limits of time and the parameter u which was introduced above takes the place of ($-\omega$). Equation 7 is the analysis equation; it obtains the space factor from the aperture illumination. By the reciprocal properties of Fourier transforms it is immediately possible to write the synthesis equation

$$I(x) = \frac{X}{\pi} \int_{-\infty}^{\infty} S(\omega)\, \epsilon^{-jux}\, du \tag{8}$$

Thus it can be seen that the limitations upon the space factor of an array contained wholly within a fixed space interval are similar to the limitations upon the frequency spectrum of a signal which is wholly contained within a specified time interval.

Before extending the analysis it is well to review how the above results were obtained:

(1) Arrays of identical parallel antennas may be considered as being made up of a continuous distribution of identical sources.

(2) The radiation-directivity pattern of an array is the product of the directivity pattern of the individual antennas and the geometrical space factor. The aperture illumination is defined by a function which defines the relative amplitudes and phases of the currents or voltages fed to the aperture. The sum or resultant of the contributions of all parts of the array are combined in the space factor which takes the form of a Fourier integral.

* In this report "spectrum" describes the amplitude and phase of the Fourier Transform, rather than amplitude squared.

(3) The directivity pattern of an array is a phase shift phenomenon, much like the selectivity of a resonant circuit. It is useful at this point to recall some of the properties of Fourier transforms and the relations between time pulses and frequency spectra, but to utilize, instead of frequency and time, the u and x coordinates which we use here. For example, Figure 5a shows one form which an aperture illumination might take and Figure 5b shows its spectrum relative to the new coordinate, u. Both the illumination and the spectrum can have real and imaginary components. However, it may be recognized that only certain values of u correspond to real values of azimuth and elevation.

This will be discussed in greater detail later when a broader basis for discussion has been developed.

It was stated earlier that a linear array cannot produce any arbitrary pattern over the entire surface of the radiation sphere. Therefore, in order to show clearly the physical significance of the spectrum of the aperture illumination it will be necessary to consider the simplest type of array which has sufficient generality such that it can theoretically produce any pattern; this is the plane area array.

THE TRANSFORM PLANE

A plane array of antennas is an array in which all the antennas are identical and parallel and the centers all lie in a plane, for example the horizontal plane. The space factor is obtained by assuming the antennas isotropic. An array of this type is shown in Figure 1b and if rectangular coordinates are used to represent distance in the plane then it is convenient to consider the array as wholly contained within some rectangle, such as the one which is

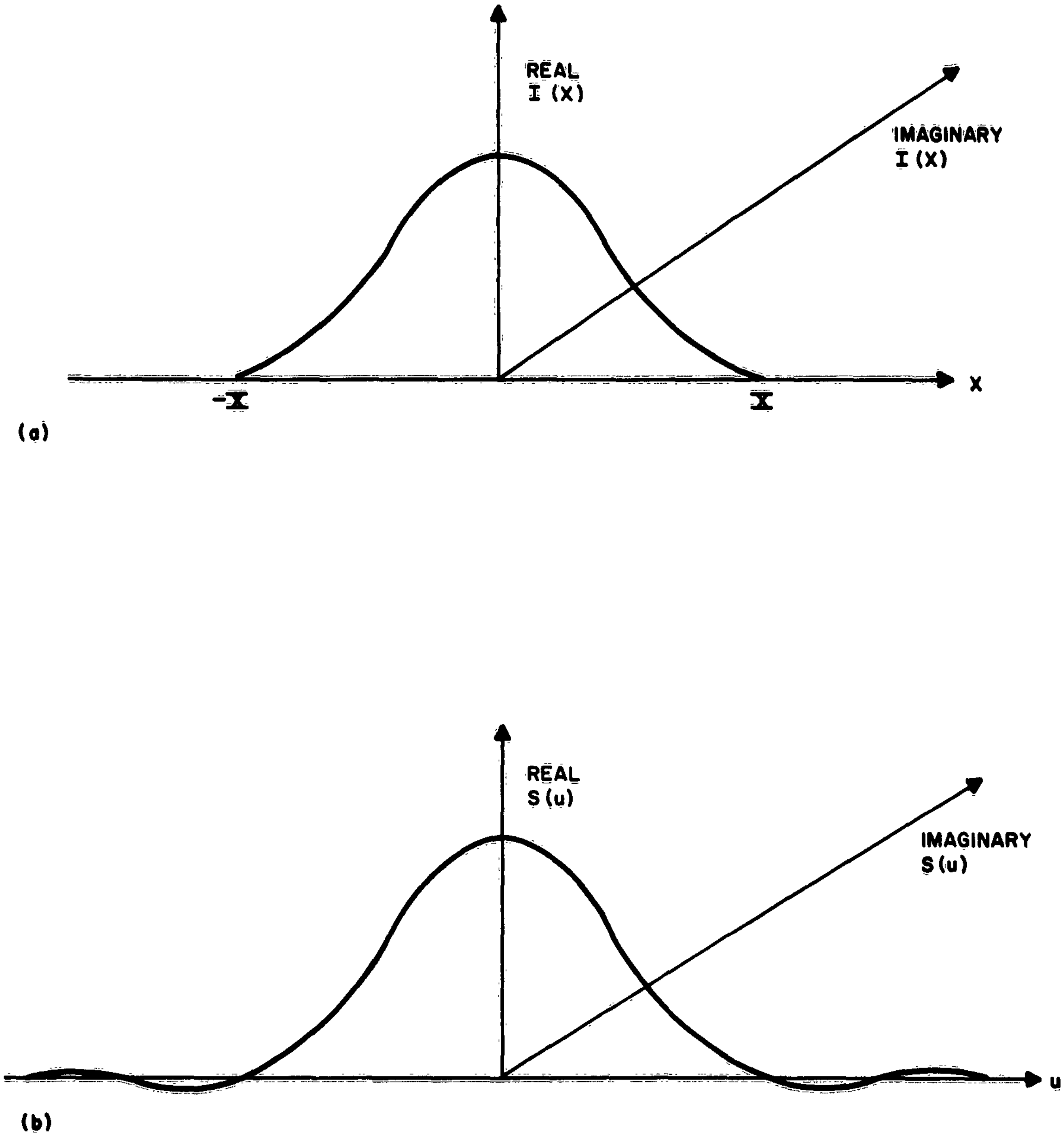


FIG. 5

shown. In this case phase shifts are also introduced by the y coordinates of the positions of particular antennas as compared with an antenna at the origin. For example the space phase shift between an antenna at position 3 and an antenna at position 1 is expressed by

$$\Delta\phi_{31} = x_3 \cos\alpha \cos h + y_3 \sin\alpha \cdot \cos h \tag{9}$$

The phase of the voltage vector may be made to be zero for a wave coming from the direction (α_o, h_o). A new parameter v is defined as shown below:

$$v \equiv \sin\alpha \cos h - \sin\alpha_o \cos h_o \tag{10}$$

The aperture illumination must now be represented as a function of both x and y; hence when the sum of the contributions of all the currents in all the parts of the array is found by integration the spectrum appears in the following form:

$$S(u,v) = \frac{1}{4XY} \int_{-X}^{X} \int_{-Y}^{Y} I(x,y)\, \epsilon^{j(ux+vy)}\, dy\, dx \tag{11}$$

Equation 11 is known as a double Fourier integral. As an example of a double Fourier integral, a two-dimensional pulse, I(x, y) which is transformed into a two-dimensional spectrum, S(u, v), is shown in Figure 6. This shows the case where both pulse and spectrum are real functions. The spectrum of the aperture illumination is a function of the two variables u and v and is represented in magnitude by a surface over the (u, v) plane. (Another surface over the (u, v) plane could be used to represent phase.) The (u, v) plane is called the transform plane and has considerable physical significance. Any type of aperture illumination results in a particular type of pattern in the transform plane.

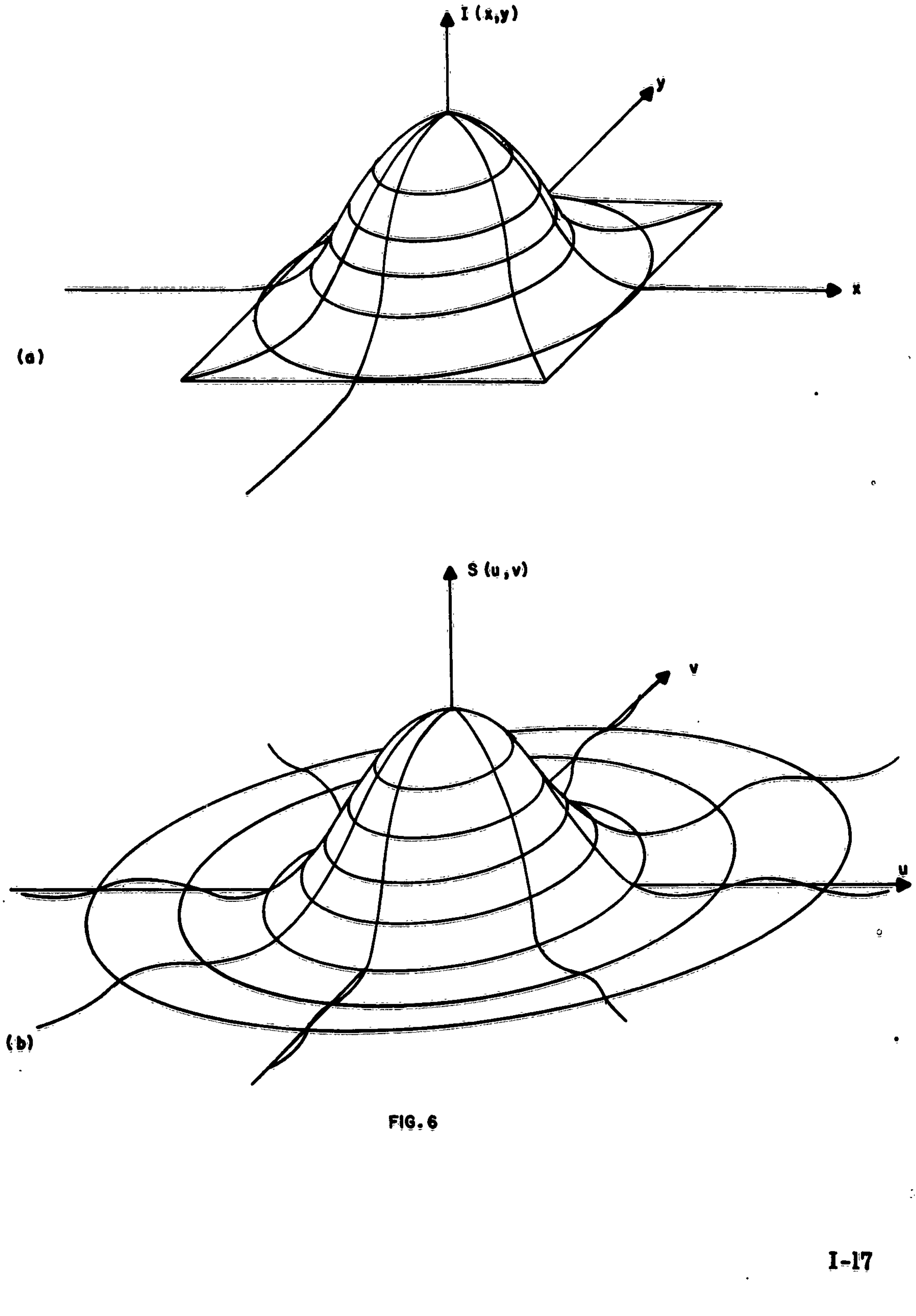
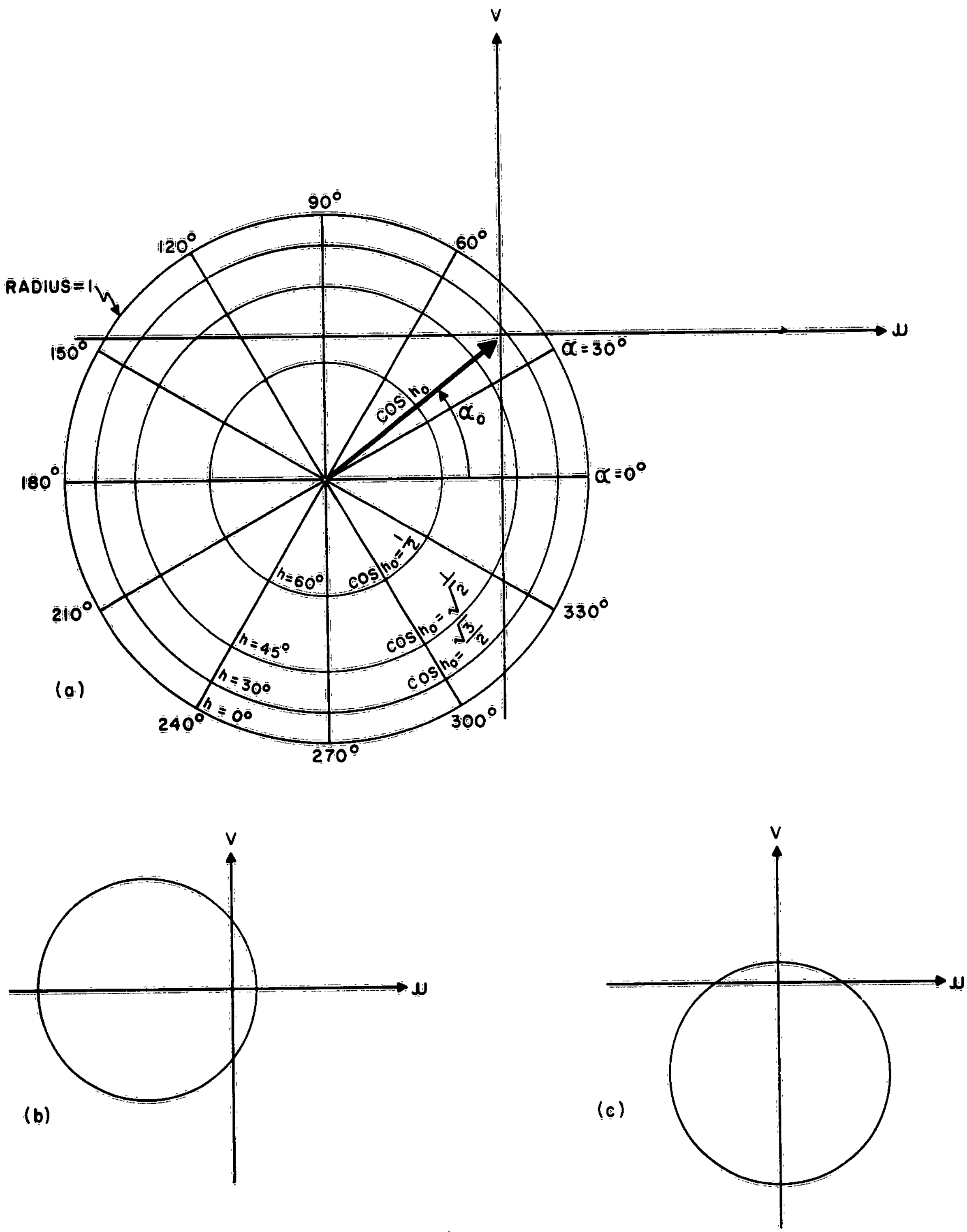


FIG. 6

The space factor at any point on the surface of the radiation sphere can be evaluated if we now find the relation between points in the transform plane and the corresponding points on the radiation sphere. This relation will, of course, depend on whether the plane of the array is horizontal, vertical or oblique. For a horizontal plane array the equations of the parameters u and v show that radiation space, as defined by the points on a hemisphere, maps into a circle in the transform plane such as is shown in Figure 7a. This circle will be seen to be identical with that previously shown in Figure 2a. Because the direction of zero azimuth was taken as parallel to the x axis in the plane of the array it appears in the transform plane parallel to the u axis. The circle has a unit radius. The distance from the center of the circle to the origin is cos h_o. The origin corresponds to the direction (α_o, h_o). This leads to a significant physical conclusion. The pattern in the transform plane is completely determined by the aperture illumination I(x, y). The location of the points in the transform plane which correspond to real values of azimuth and elevation are completely and independently determined by the initial phase adjustment. The value of the spectrum at any point in the transform plane which corresponds to a real direction in radiation space is the space factor for that direction and has the proper phase angle.

With the definitions of the parameters given earlier in equations 6 and 9 the circle corresponding to radiation space (or the radiation circle) can occupy any position in the transform plane such as to include the origin within or on its circumference. In a more general case the radiation circle can appear anywhere in the transform plane. This is useful for line-source arrays.

The positions of the radiation circle for crossfire, endfire, and broadside arrays are shown in Figures 7a, b and c.



V
U
90°
120°
60°
RADIUS=1
150°
α=30°
COS h₀
α₀
180°
α=0°
h=60°
COS h₀=1/2
COS h₀=1/√2
COS h₀=√3/2
h=45°
h=30°
h=0°
210°
330°
240°
300°
270°
(a)
(b)
(c)

FIG. 7

Examples of patterns obtained in analysis or synthesis problems of plane arrays will be shown later in order to clarify and extend the concepts thus far developed. However, before showing these examples it is useful and informative to inquire further into the reason why the radiation hemisphere maps onto the transform plane in the particular way that it does. In order to do this it is convenient to consider the radiation pattern of the most general type of array, in which the antennas are distributed throughout a volume.

TRANSFORM SPACE

Two antennas of a volume array are shown in Figure 1c. In this case the relative space phase shift of antenna #4 with respect to antenna #1, including the phase adjustment which brings all the vectors in phase for one direction is included in

$$\epsilon^{j(ux+vy+wz)} \qquad (12)$$

where

$$w = \sin h - \sin h_0 \qquad (13)$$

In this case the spectrum must be represented by a volume, or three dimensional space, called transform space. The locus of points corresponding to the various azimuth and elevation directions may be mapped onto the surface of a unit sphere, called the radiation sphere, and which is shown in Figure 8. The sphere may be made identical with the sphere shown in Figure 2. The direction from the center of the sphere to the origin is (α_0, h_0) and the origin is a point on the surface of the sphere. For some applications, it is desirable to have the

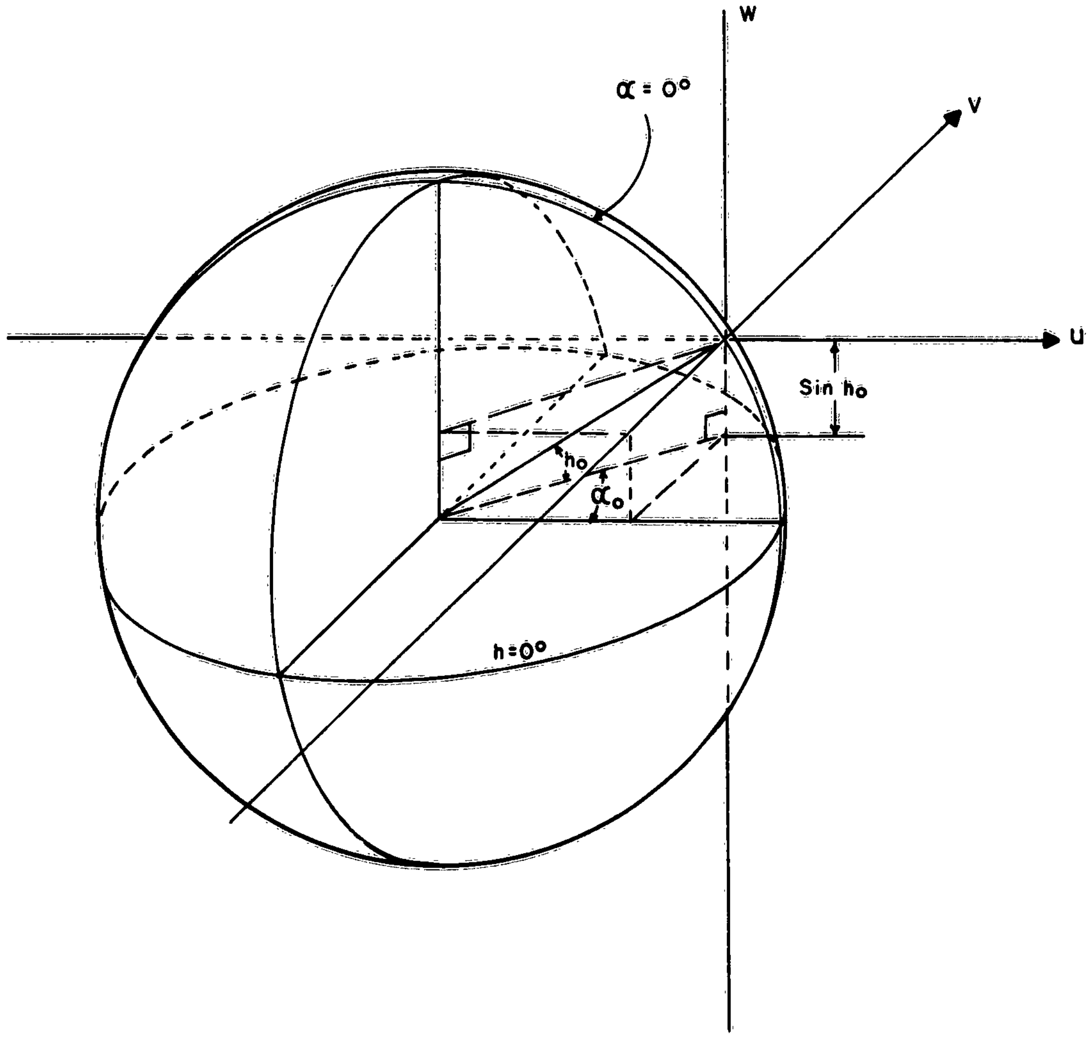


FIG. 8

origin outside the radiation sphere. In this case, parameters γ_u , γ_v and γ_w are added to u, v, and w respectively. The radiation sphere, as a function of u and v, may be completely defined by its orthographic projections. In the case of a plane array, which is represented in actual space by only two dimensions, the patterns will be a function of only two of the three transform coordinates.

Every point in transform space has associated with it an intensity, specified by the space factor, S, which is a function of the three variables u, v and w.

This is like an electric field for which the intensity is specified at every point. The dimensions of S are complex field intensity, defined by an amplitude and a phase.

The amplitude and phase associated with those points in transform space which coincide with the surface of the radiation sphere define the radiation for the corresponding values of azimuth and elevation. This is the reason why this method of representation was chosen earlier.

SPECTRA OBTAINABLE FROM RECTANGULAR PLANE ARRAYS

It is intended that the mathematical treatment of the radiation theory herein presented shall be left for later parts of this report. However, it is useful at this time to make use of one of the relationships which will be developed later in order to justify and explain some of the physical examples and sketches which will be shown here to indicate the physical limitations upon the spectra of aperture illuminations and the patterns which may be obtained in radiation space. For example, it is convenient to describe a typical aperture illumination.

An illustrative continuous aperture illumination for an array contained within a rectangle of area $4XY$ was shown in Figure 6. The illumination is conveniently expressed mathematically in terms of a Fourier series, as is required for a two dimensional function. It is useful to write the double Fourier series in exponential form as shown below because when this form is substituted into equation 11 and integrated a particularly useful form is obtained.

$$I(x,y)=\sum_{M=-\infty}^{\infty}\sum_{K=-\infty}^{\infty} I_{KM}\, \epsilon^{j\left(K\pi\frac{x}{X}+M\pi\frac{y}{Y}\right)} \tag{14}$$

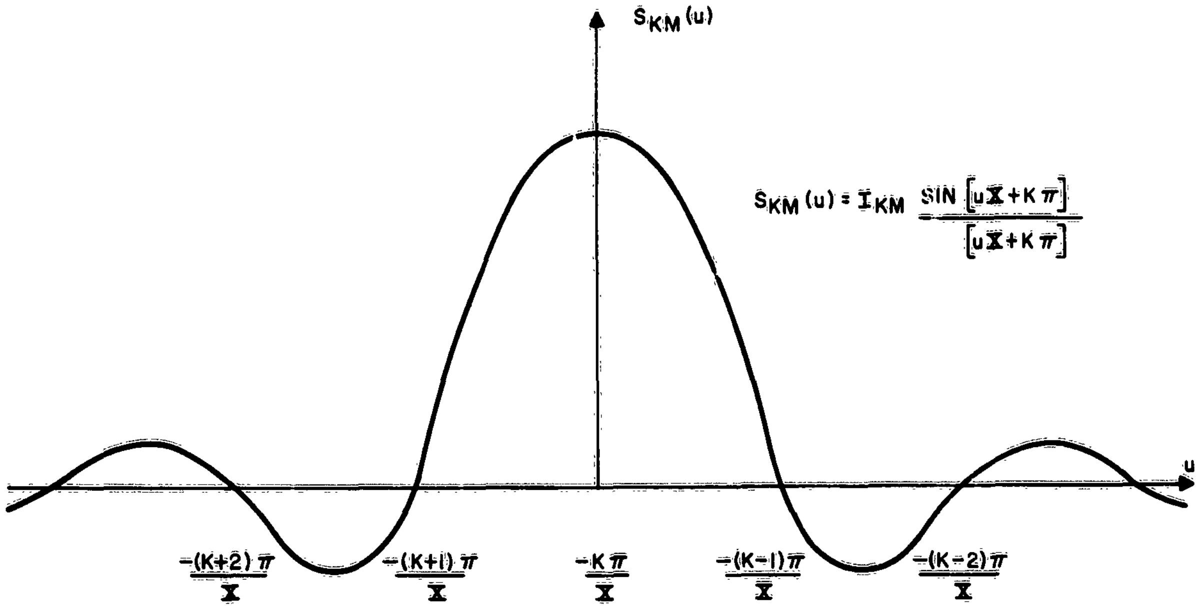
For simplicity, consider a single harmonic term of the series such as

$$I_{KM}\, \epsilon^{j\left(K\pi\frac{x}{X}+M\pi\frac{y}{Y}\right)} \tag{15}$$

When this is substituted into equation 11 it is found that

$$S_{KM}(u,v)=I_{KM}\cdot\frac{\sin(uX+K\pi)}{uX+K\pi}\cdot\frac{\sin(vY+M\pi)}{vY+M\pi} \tag{16}$$

The spectrum, or space factor shown in equation 16 has a familiar form when either u or v is constant, since in that case it varies as the well known $\frac{\sin x}{x}$ curve. For example if $(vY+M\pi)$ is zero, then $S_{KM}(u)$ may be represented as shown in Figure 9a. This function goes to zero at intervals of u spaced by $\frac{\pi}{X}$ except that $S_{KM}(u)$ has the amplitude I_{KM} at the point where the numerator and denominator of $S_{KM}(u)$ both go to zero. This occurs when $uX + K\pi = 0$. A series of such curves with different values of K would be spaced so that the zero


$S_{KM}(u)$
$S_{KM}(u) = I_{KM} \dfrac{\sin[u X + K\pi]}{[u X + K\pi]}$
u
$\dfrac{-(K+2)\pi}{X}$
$\dfrac{-(K+1)\pi}{X}$
$\dfrac{-K\pi}{X}$
$\dfrac{-(K-1)\pi}{X}$
$\dfrac{-(K-2)\pi}{X}$

(a)

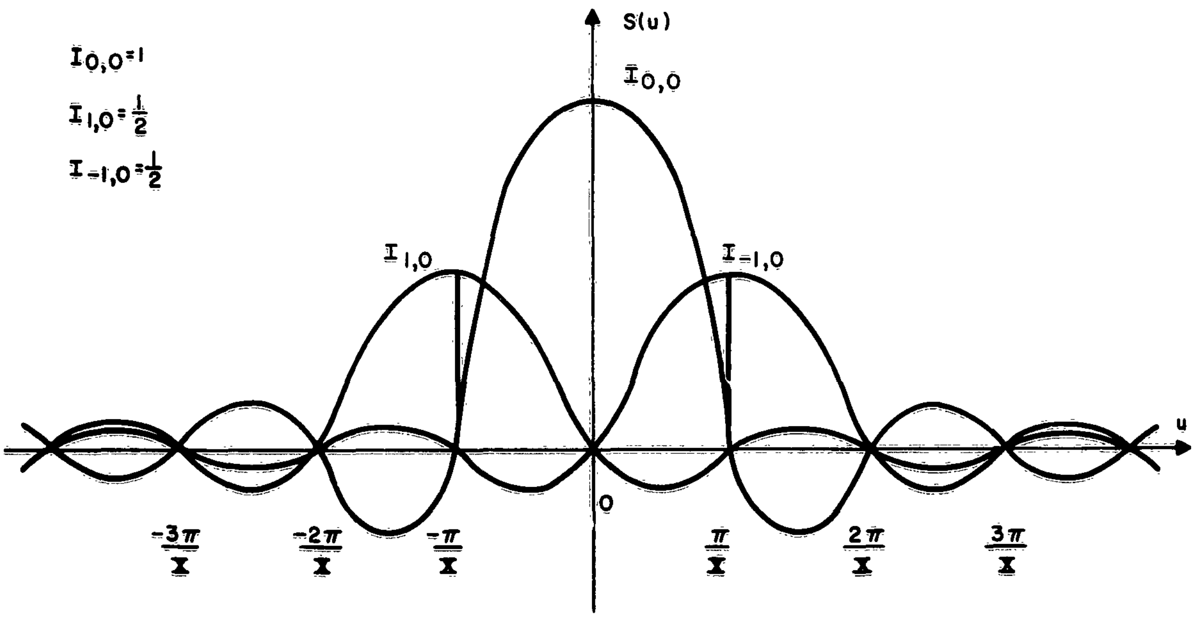

S(u)
$I_{0,0}=1$
$I_{1,0}=\frac{1}{2}$
$I_{-1,0}=\frac{1}{2}$
$I_{0,0}$
$I_{1,0}$
$I_{-1,0}$
u
0
$\frac{-3\pi}{X}$
$\frac{-2\pi}{X}$
$\frac{-\pi}{X}$
$\frac{\pi}{X}$
$\frac{2\pi}{X}$
$\frac{3\pi}{X}$

FIG. 9

(b)

points coincide and therefore a series of three terms such as shown in Figure 9b is completely specified by the amplitudes $I_{1,o}$, $I_{o,o}$, and $I_{-1,o}$. The points where the zeros can occur are called the independent points. A function which is known to be made up of a series of this type of $\frac{\sin x}{x}$ functions may be completely specified by its values at the independent points, which (it may be noted) are equal to the coefficients in the Fourier series for the aperture illumination. In the two dimensional case where S (u, v) is given by equation 16, the spectrum may be represented by a 3-dimensional sketch such as was shown in Figure 6, or, alternatively and more simply, by the locations and amplitudes of the independent points as indicated in Figure 10.

It can be seen that for any aperture illumination which can be expressed as a Fourier series having only a small number of low order harmonics the I_{KM} coefficients will be different from zero only at those independent points which are near the origin.

THE SYNTHESIS METHOD

The synthesis method consists of building up the desired pattern out of a set of standard building blocks or eigenfunctions. These eigenfunctions result from taking the Fourier transform of the aperture illumination when the illumination is expressed as a suitable Fourier series. Part II of this report gives the details of suitable eigenfunction expansions for several classes of arrays.

As an example of synthesis consider a rectangular plane array for which the eigenfunctions have the $\frac{\sin x}{x}$ form shown above. Suppose it is desired to design such an array having a single narrow main lobe and virtually no side lobes. It is useful in this case to use a particular set of the transform pulses known as the cosine squared type. Cosine squared pulses have been used to produce a flat field on a television screen. This subject is discussed by A. V. Loughren

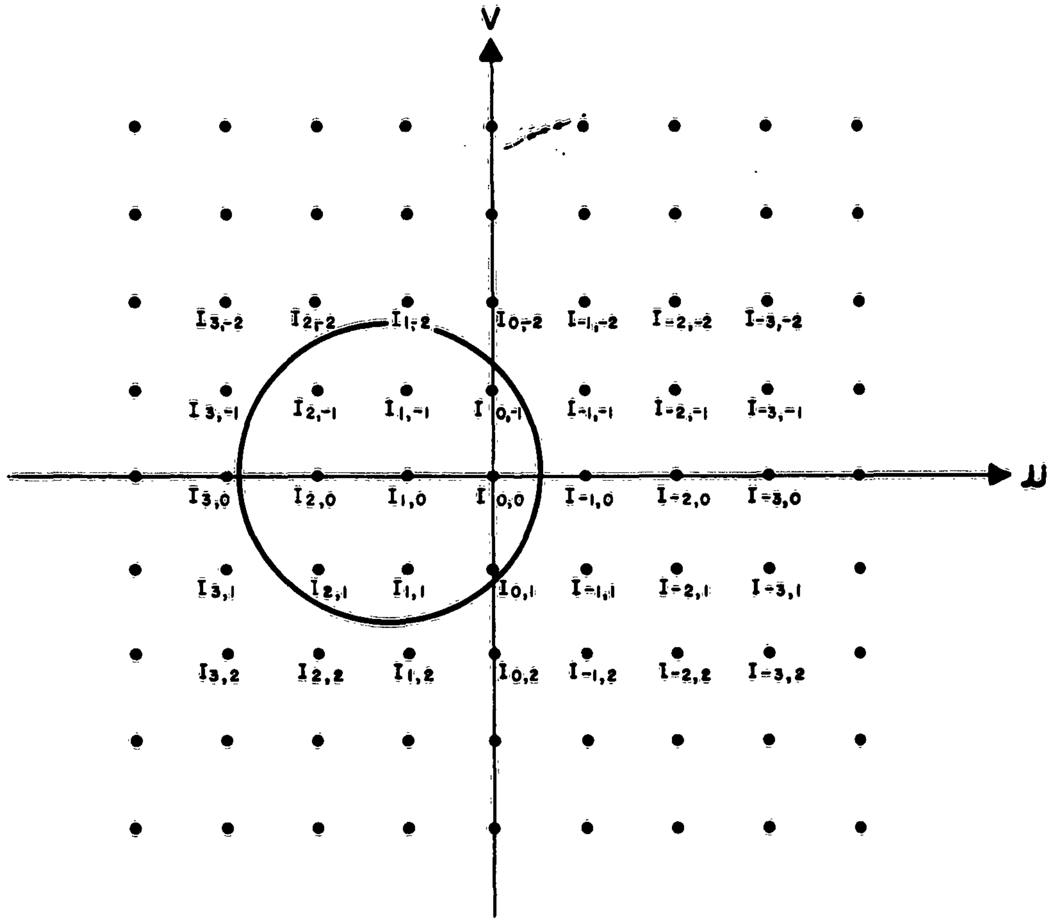

V
U
I3,-2 I2,-2 I1,-2 I0,-2 I-1,-2 I-2,-2 I-3,-2
I3,-1 I2,-1 I1,-1 I0,-1 I-1,-1 I-2,-1 I-3,-1
I3,0 I2,0 I1,0 I0,0 I-1,0 I-2,0 I-3,0
I3,1 I2,1 I1,1 I0,1 I-1,1 I-2,1 I-3,1
I3,2 I2,2 I1,2 I0,2 I-1,2 I-2,2 I-3,2

FIG. 10

and H.A. Wheeler in "The Fine Structure of Television Images," Proceedings of I.R.E., May 1938. * A cosine squared pulse is indicated in Figure 5a; it has a Fourier transform which is very close to being a cosine squared pulse also. This is shown in Figure 5b. The transform of the cosine squared pulse can be defined by the set of coefficients.

$$I_{0,0} = 1 \qquad I_{1,0} = I_{-1,0} = \frac{1}{2} \tag{17}$$

Figure 9 was drawn in these proportions. In the two dimensional case a conveniently shaped surface may be built up by using the set of coefficients shown in Figure 11a. The space factor thus produced has virtually no side lobes (the first side lobe being 32 db down). The space factor which is thus obtained was indicated earlier in Figure 6b. The space factor for $\alpha_0 = 0$ and $h_0 = 0$ was sketched in Figure 2b. The required aperture illumination has the following form, other than the initial phase adjustment of α_0 and h_0 to direct the major lobe in the desired direction.

$$I(x, y) = 4 \cos^2\left(\frac{\pi x}{X}\right) \cos^2\left(\frac{\pi y}{Y}\right) \tag{18}$$

and was sketched in Figure 6a. This is one of the simpler examples of the use of a shaped aperture illumination to obtain a specific directivity pattern.

If the aperture had been uniformly illuminated then $I(x, y) = I_{0,0}$ and $S(u, v) = S_{0,0}$, and the pattern would have been that of equation 16 with K and M set equal to zero. The side lobes for uniform illumination are much larger than for cosine-squared illumination, particularly in the u = o and v = o directions. In these cases the maxima are the same as in Figure 9.

* Hazeltine Electronics Corporation Report #771 BW.

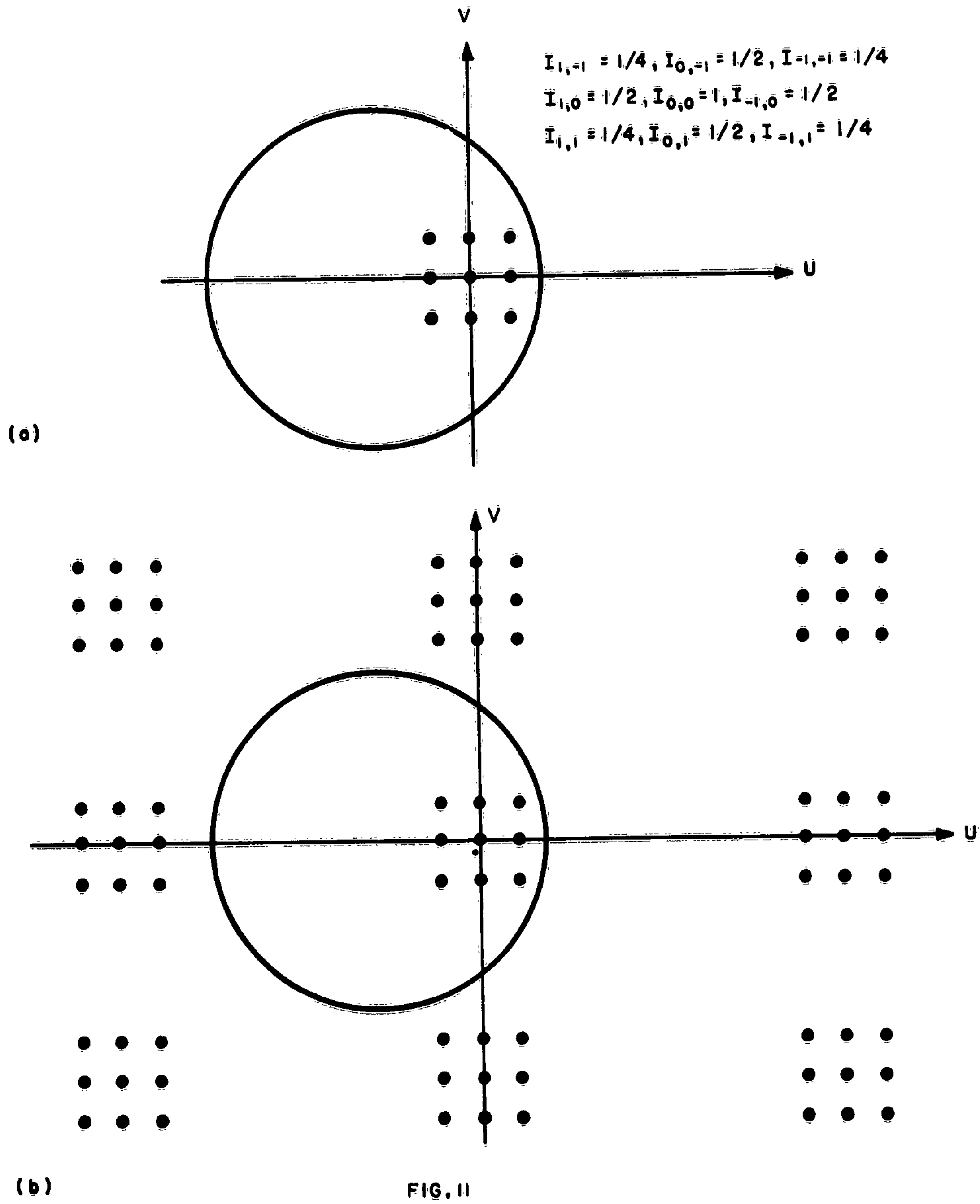

V
$I_{1,-1} = 1/4, I_{0,-1} = 1/2, I_{-1,-1} = 1/4$
$I_{1,0} = 1/2, I_{0,0} = 1, I_{-1,0} = 1/2$
$I_{1,1} = 1/4, I_{0,1} = 1/2, I_{-1,1} = 1/4$
U
(a)
V
U
(b)

FIG. 11

The synthesis problem in the simplest case is approached by building up the desired pattern from the basic building blocks and then finding the necessary aperture illumination. The design of arrays which make economical use of the available space and which utilize only a small number of radiating elements in the array requires the careful consideration of several factors which are discussed in detail in Section III of this report. These factors are:

(1) The type of radiating element and the orientation of the radiating elements with respect to the array which results in the most useful function $F_o(\alpha, h)$.

(2) The combinations of eigenfunctions which reproduce the desired pattern to a satisfactory approximation consistent with an acceptable compromise between size of aperture and density of antennas.

(3) The modification of the pattern of a continuous illumination by discrete antennas, and the most advantageous use of this effect.

ARRAYS OF DISCRETE ANTENNAS

An array of discrete antennas may be analyzed or synthesized by considering such an array as a continuously illuminated aperture having an illumination which is the product of two parts. One part $I_o(x)$ is similar to the illumination already considered. For example, see Figure 12a for a line-source array. The other part $P(x)$ is defined as zero except at the points occupied by the antennas. See Figure 12b. The total illumination, $I(x)$, is the product of the two as shown below:

$$I(x) = I_o(x) \quad P(x) \tag{19}$$

See Figure 12c.

The spectrum corresponding to a continuously illuminated aperture such as shown in Figure 12a is shown in Figure 13a in which a typical $\frac{\sin uX}{uX}$ term is sketched as well as the resultant of all such terms, and in which the harmonic amplitudes

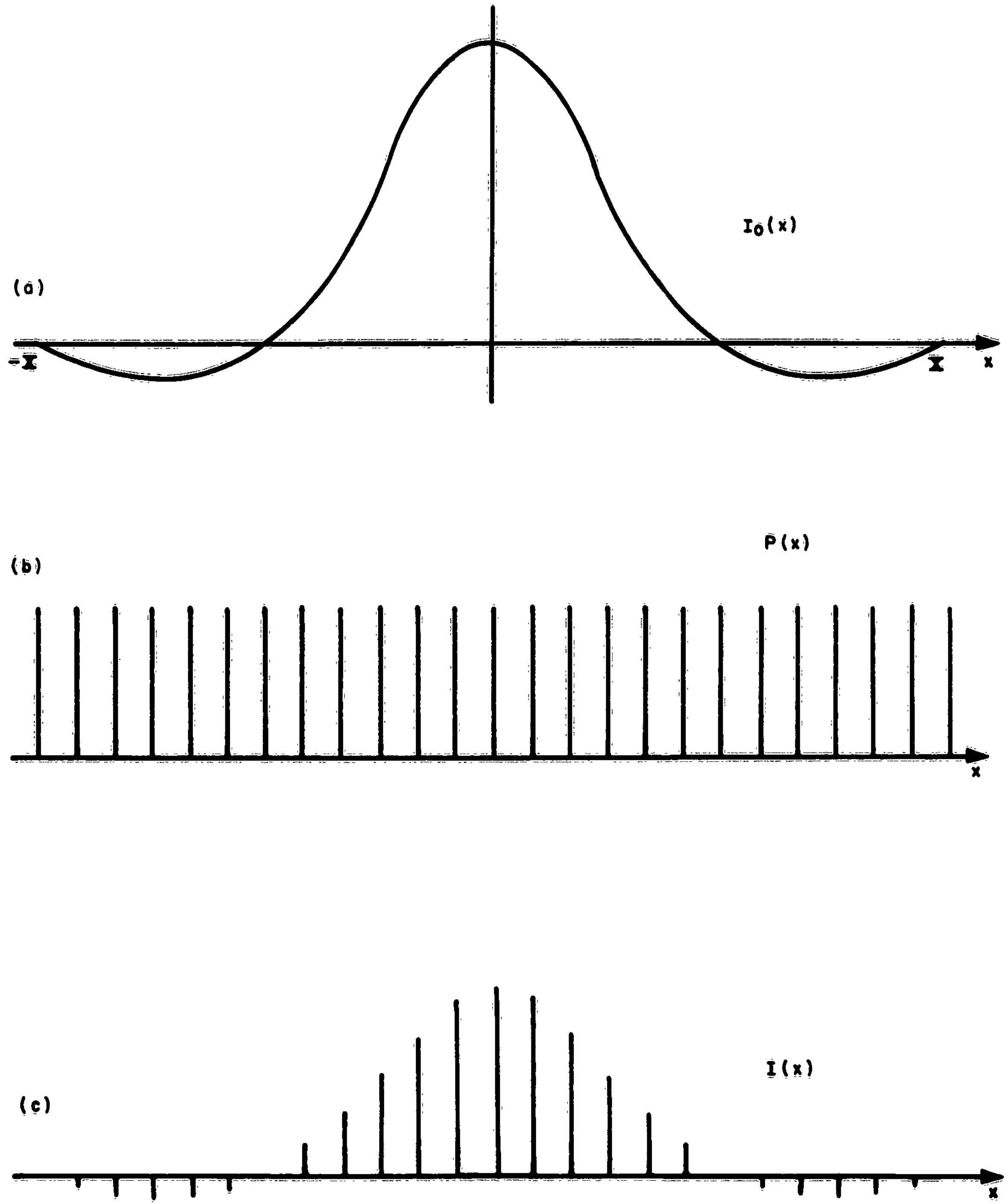


FIG.12

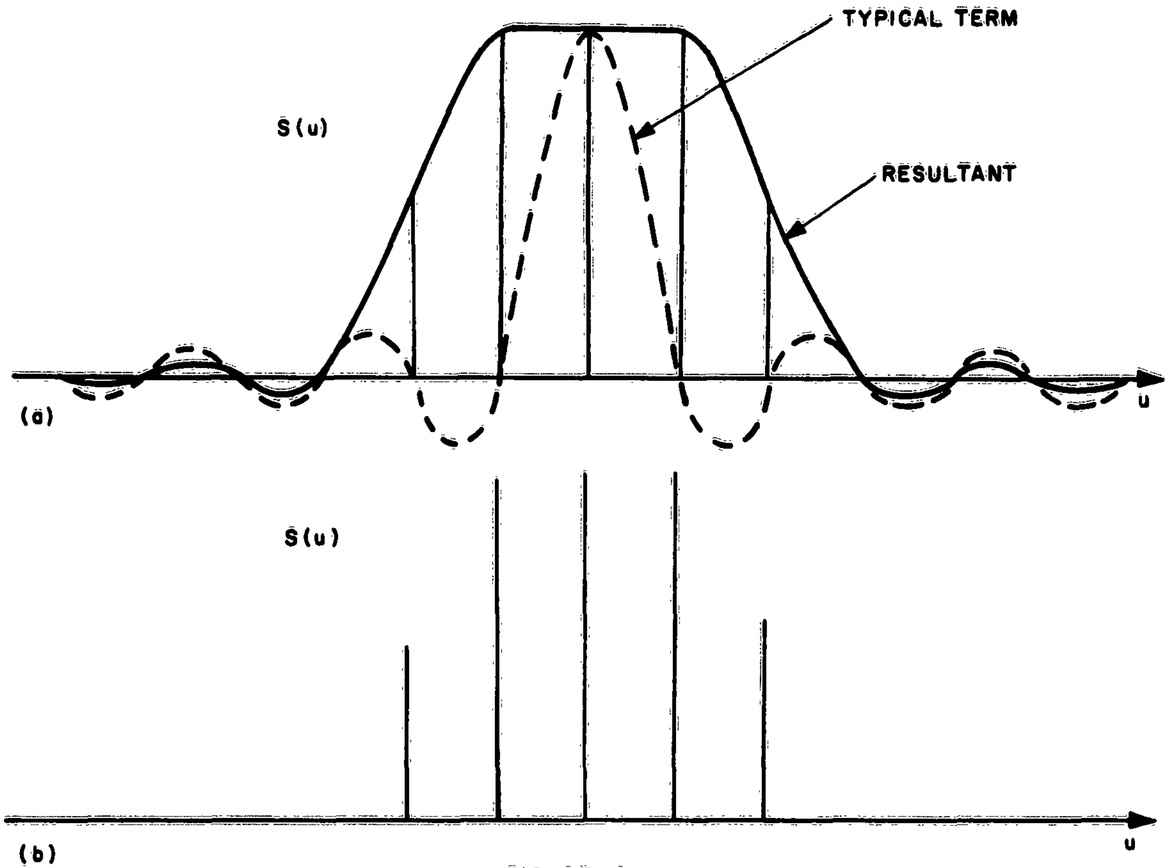


FIGURE 13

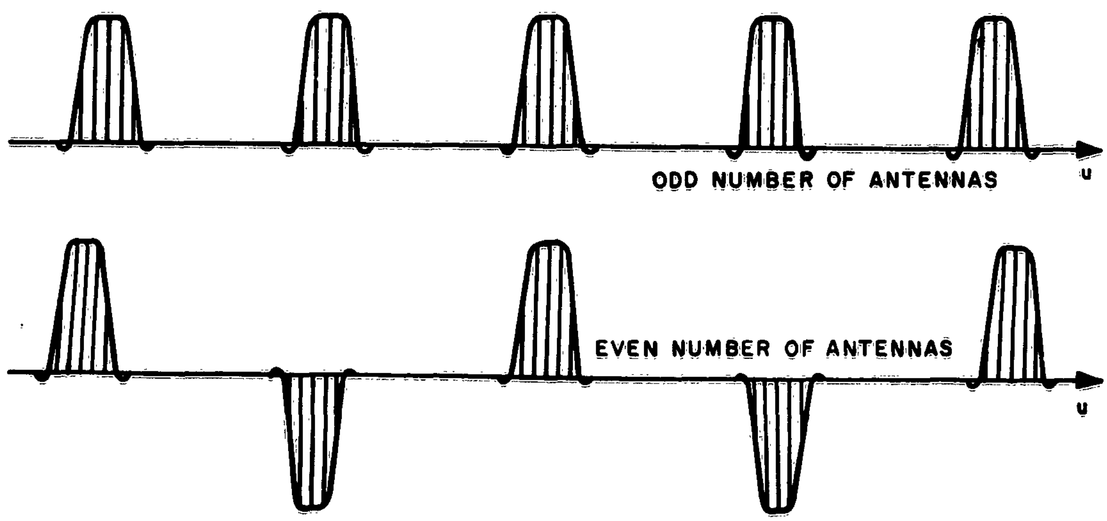


FIG. 14

are indicated. The reason the I_K coefficients in the $S(u)$ function are called the harmonic amplitudes is made clearer by consideration of Figure 13b which shows the spectrum which would be obtained if the continuous aperture illumination of length 2X which is shown in Figure 12a were repeated infinitely at intervals of 2X. This is the spectrum of a repeating pulse sequence and each vertical line in Figure 13b represents one harmonic component. The effect of the finite aperture is to modify Figure 13b to Figure 13a, which is a continuous spectrum, but still may be regarded as being made up of the harmonic components.

The effect of approximating the continuous finite aperture of Figure 12a with the discontinuous finite aperture of Figure 12c is to modify the spectral pulse of Figure 13a to the sequences of spectral pulses of Figure 14. The pulses in Figure 14 are separated from each other by a number of independent points, or harmonics, equal to the number of antennas in the particular P(x) function. This is a result of the multiplication of the two harmonic series.

These results are of a familiar form. Since the aperture illumination is like a time-pulse which is sampled at a high rate (such as for example, in a super-regenerative sampling process), the result is that the spectrum of the sampled pulse consists of a set of modulated carriers. Each carrier is a harmonic of the sampling pulse rate and each carrier when modulated is replaced by a harmonic sequence identical in form with the spectrum of the unsampled pulse but with the spectral groups centered on the carriers as is indicated in Figure 14. The highest unambiguous modulation rate is one-half the sampling rate.

It may be noted in Figure 14 that a different picture is drawn depending on whether an odd number or an even number of antennas defined the sampling rate. This is because the side lobes of the $\frac{\sin x}{x}$ function alternate in phase.

The inversion shown in Figure 14 indicates that the phase of the modification of the original pattern which results from the fact that a discrete number of antennas is used instead of a continuous illumination always has the same phase regardless of the number of antennas in the line, although the amplitude of the correction term at any point tends to decrease as the number of antennas increases.

In the case of a rectangular plane area array the pattern shown in Figure 11a, in which only nine spectral components appear for a continuous illumination, must be modified to the pattern shown in Figure 11b in which the nine-sequence is repeated over the entire plane in a rectangular lattice in which the spacing in the u direction is equal to the number of antenna rows along the x axis and the spacing in the v direction is equal to the number of antenna rows along the y axis.

The number of antennas needed within the specified aperture can be found from Figure 11 since it is merely necessary to insure that the extra lobes which result from the discreteness of the antennas do not undesirably alter the pattern or space factor in that portion of the transform plane which corresponds to radiation space.

This problem will be discussed more fully in Section III of this report. However, in order to round out the general picture of the theory herein presented an example will be given next of an analysis problem to which the theory was applied.

ANALYSIS OF A CIRCULAR LINE ARRAY OF DISCRETE ANTENNAS

There are a large number of forms in which the formula for the pattern of an array may be written. Some of these involve a reasonably small number of simple terms. Others may have sequences of infinite series which must be

added to give the same results as the closed form solutions. For each general shape of aperture there is a particular set of coordinates, suited to the geometry of the aperture, which is most convenient for representing both the aperture illumination functions and the spectra. In the case of arrays of circular shape, one example of which is the array of antennas on the perimeter of a circle in a horizontal plane considered here, it is convenient to use polar coordinates such as are shown in Figure 15a to represent the geometry of the antenna array. The angle β is used to represent angular position of an antenna from the x axis, as shown, and the radius r represents the distance from the origin to the particular antenna. The phase of the voltage induced in any antenna on a circle, compared to that which would be induced in an antenna at the center is $\Delta\phi = \rho r \cos(\beta - \theta)$ Two new parameters, ρ and θ appear here. This form could be obtained directly from the earlier form for rectangular coordinates by a standard transformation of the variables; ρ and θ may be defined in terms of u and v.

In this case it is found that the transform plane is most conveniently described in terms of the polar coordinates shown in Figure 15b in which θ represents the angle made with the u axis by a vector from the point $\rho = 0$ to any arbitrary point. The radial distance is ρ . The actual spectrum is independent of the set of coordinates used.

As is the case with most problems in electromagnetic theory which involve circular geometry the solutions are most conveniently obtained in closed form in terms of Bessel functions of the first kind. The general formulas for circular arrays are presented in Part II of this report. The Bessel functions of the first kind are all very much alike, as can be seen from Figure 16.

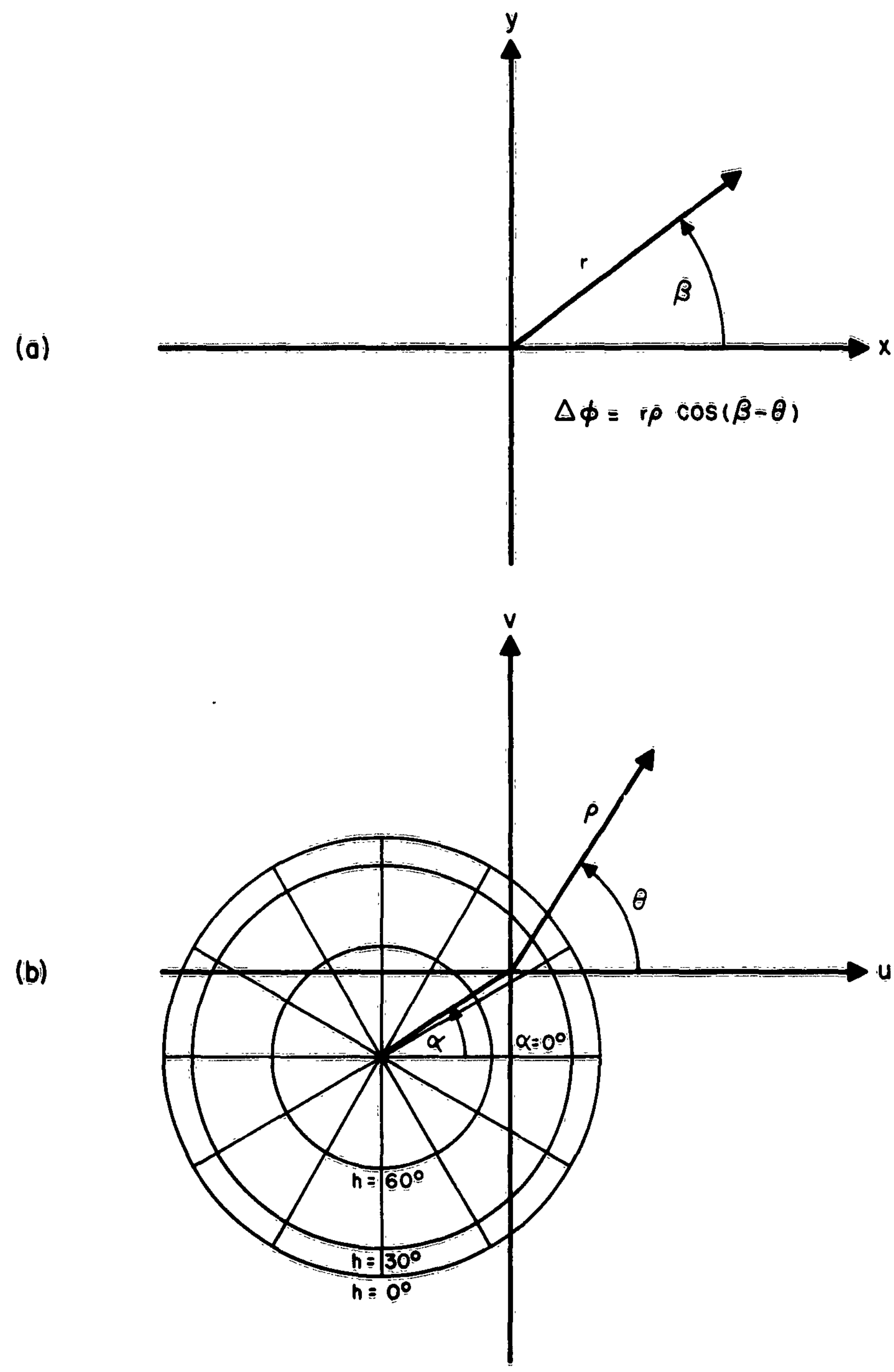


FIG. 15

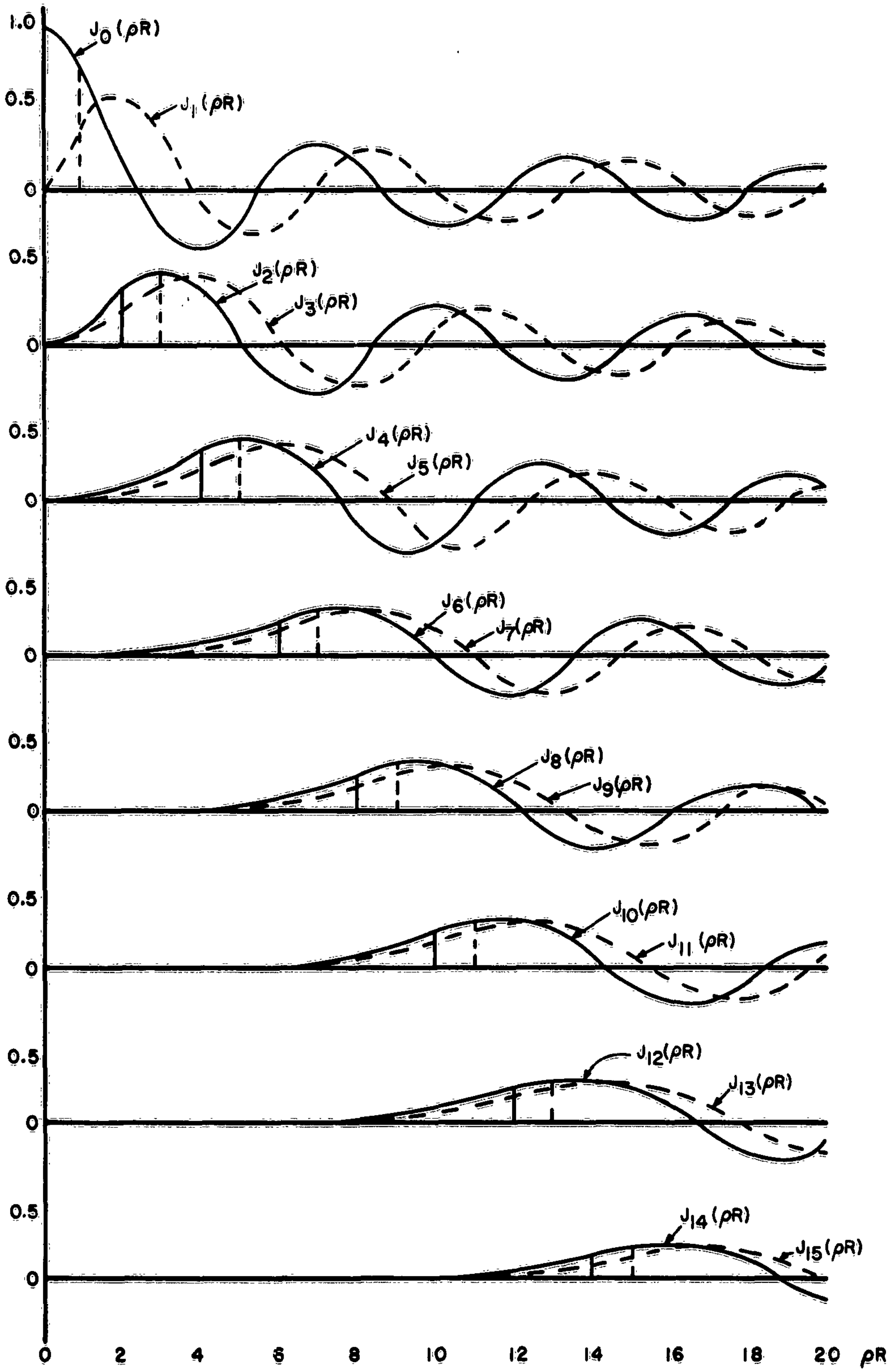

1.0
0.5
0
0.5
0
0.5
0
0.5
0
0.5
0
0.5
0
0.5
0
0.5
0
$J_0(\rho R)$
$J_1(\rho R)$
$J_2(\rho R)$
$J_3(\rho R)$
$J_4(\rho R)$
$J_5(\rho R)$
$J_6(\rho R)$
$J_7(\rho R)$
$J_8(\rho R)$
$J_9(\rho R)$
$J_{10}(\rho R)$
$J_{11}(\rho R)$
$J_{12}(\rho R)$
$J_{13}(\rho R)$
$J_{14}(\rho R)$
$J_{15}(\rho R)$
0 2 4 6 8 10 12 14 16 18 20 ρR

FIG. 16

The eigenfunctions which appear in the case of the circular array are combinations of Bessel functions of the first kind of argument ρR, and trigononometrical functions of θ. For example, an array of ten antennas spaced 1/2 wave length apart around a circle and having equal currents for all the antennas would have its spectrum expressed by the equation

$$S(\rho,\theta) = J_0(\rho R) - 2J_{10}(\rho R)\cos 10\theta \tag{20}$$

+ terms in $J_{20}(\rho R)$ and other higher terms, all of which can be neglected.

The first term in this expression represents the pattern obtainable for a continuously illuminated aperture; this pattern is sketched in Figure 17 which indicates by an isometric drawing the spectrum corresponding to each point of the transform plane in the vicinity of the origin. The second term in the expression for $S(\rho,\theta)$ shown above is the only factor introduced by the fact that the array consists of ten antennas. This factor is shown for small values of ρR, by the isometric drawing of Figure 18. Since both are real functions, the total amplitude is the sum, with due regard to sign, of the amplitudes shown in Figure 17 and Figure 18. The correction term is zero to several decimal places for a distance from the origin which increases with the density of antennas on the perimeter of the circle. For example, it may be seen from Figure 16 that the amplitude of any Bessel function is negligible until the argument is as large as a few integers less than the order, as indicated by the values of $J_k(K)$. The significance of this fact will be explained in greater detail in a later section of this report in which the mathematics of circular arrays will be discussed. However, it can be seen here that the effect of discreteness in circular arrays is similar to that already found in rectangular arrays.

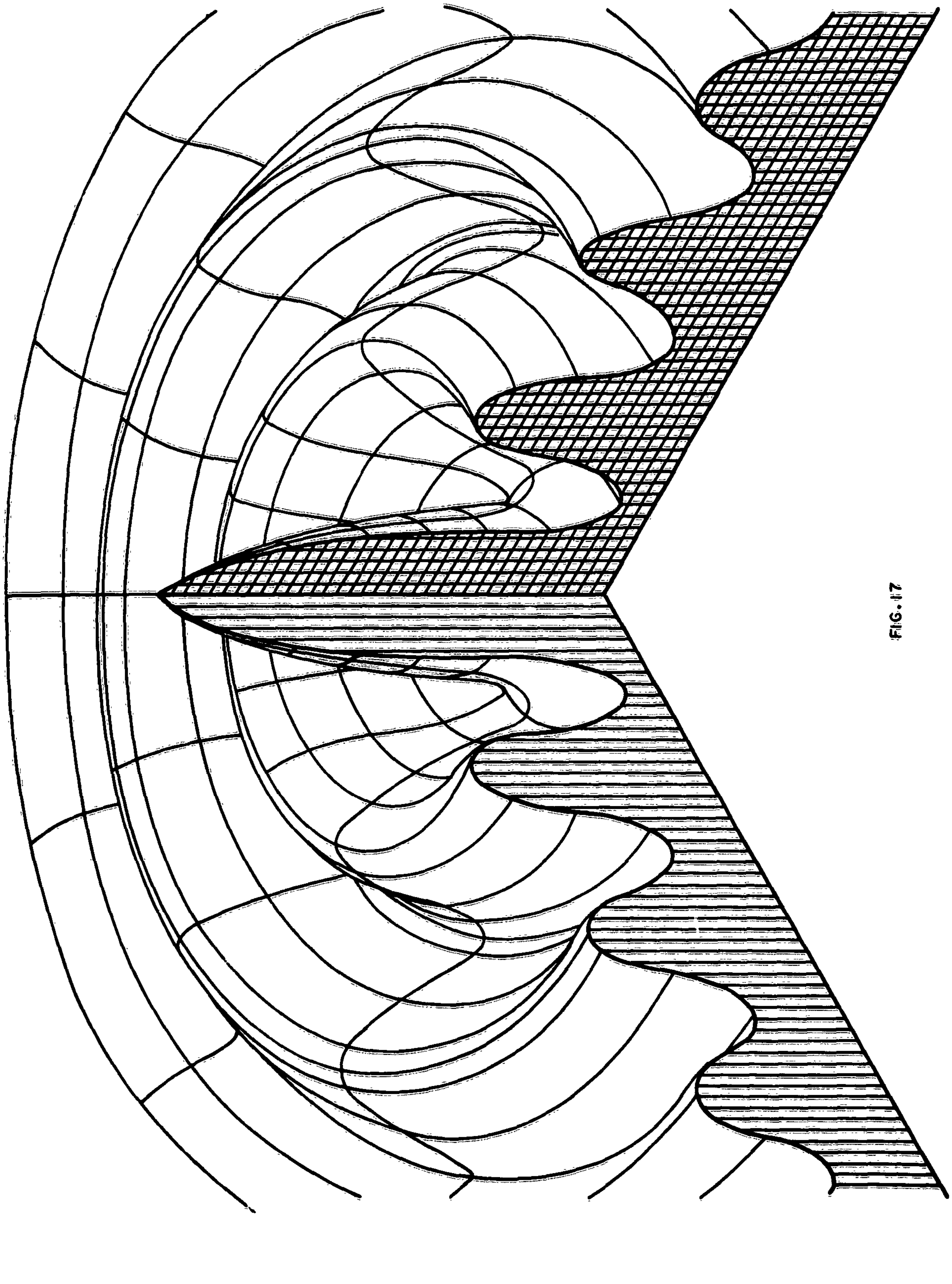


FIG. 17

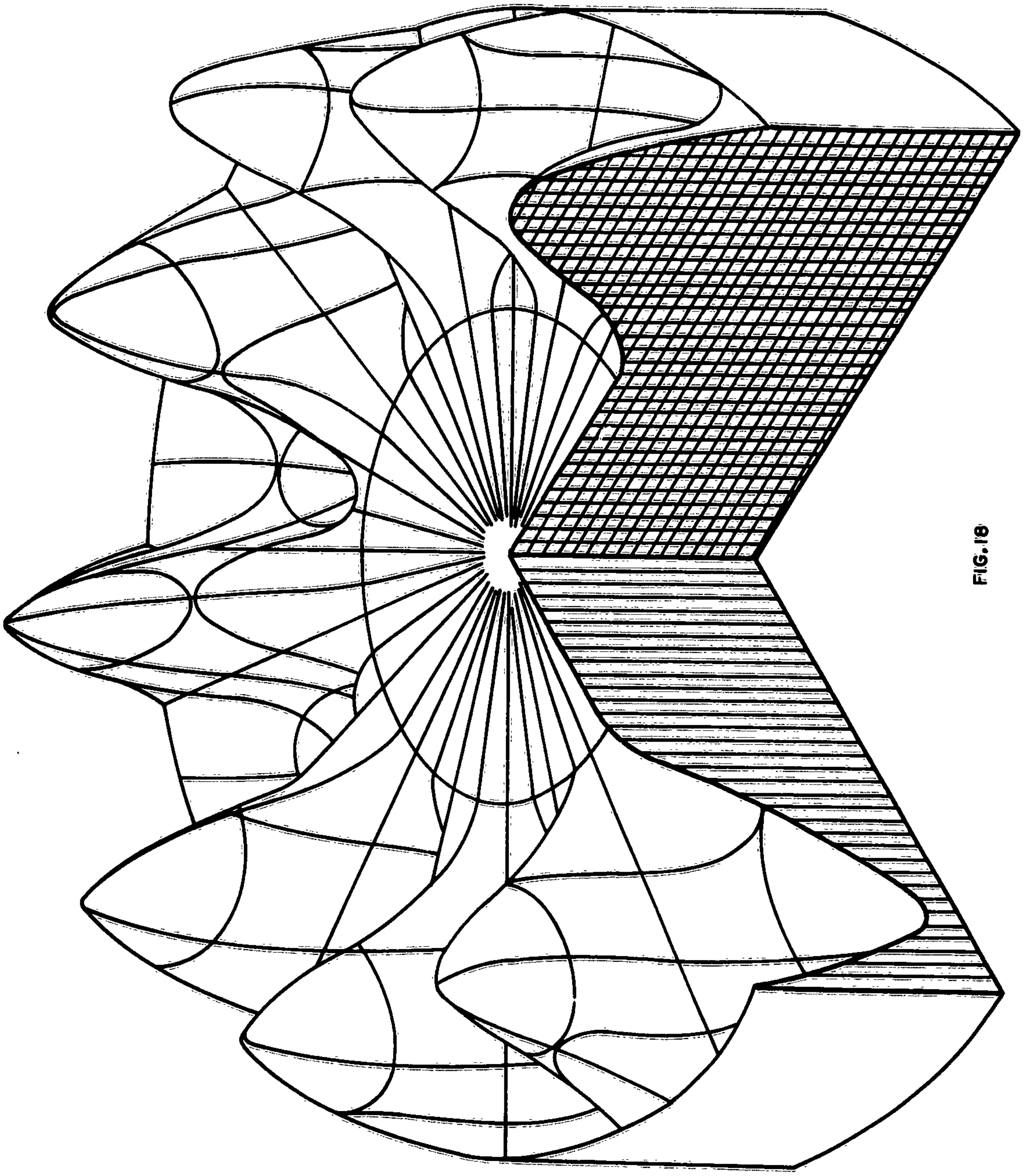


FIG. 18

The procedure used in analyzing or finding the pattern of a circular array consists of drawing on a suitable set of coordinates, (ρ, θ), lines corresponding to maxima, minima and zeros of S in the transform plane. Along any radial line θ is a constant and the spectrum is a function of a single variable, ρ. Along any circular line concentric with the origin of the transform plane ρ is a constant and the spectrum is a function of a single variable, θ; critical points on the intensity contour curves for various portions of the transform plane can thus be obtained by simple graphical means using simple functions of a single variable. This would not be possible if it were attempted to find the patterns as a function of azimuth and elevation directly.

When the pattern lines have been plotted in the transform plane they may be traced on to a piece of special graph paper which represents radiation space. Figure 19 is a sample of the graph paper used. With circular arrays α, is generally taken as zero and a choice of h, is made before tracing the pattern from the transform plane. Figure 20 shows a typical pattern for a circular array for which $\alpha_o = 0^o$ and $h_o = 30^o$. The pattern is indicated by lines of extreme or zero values of $|S|$.

Study of the graph paper shows that certain ranges of elevation are compressed; for example, it is not possible to read conveniently to within 1^o of elevation in the range 0 to 10^o of elevation. However, it will be noted that the pattern lines are approximately uniformly spaced on this paper. This is because the pattern in the transform plane is limited in the rapidity with which it can change with distance along the plane and the intensity can be read to the same accuracy for all values of azimuth and elevation.

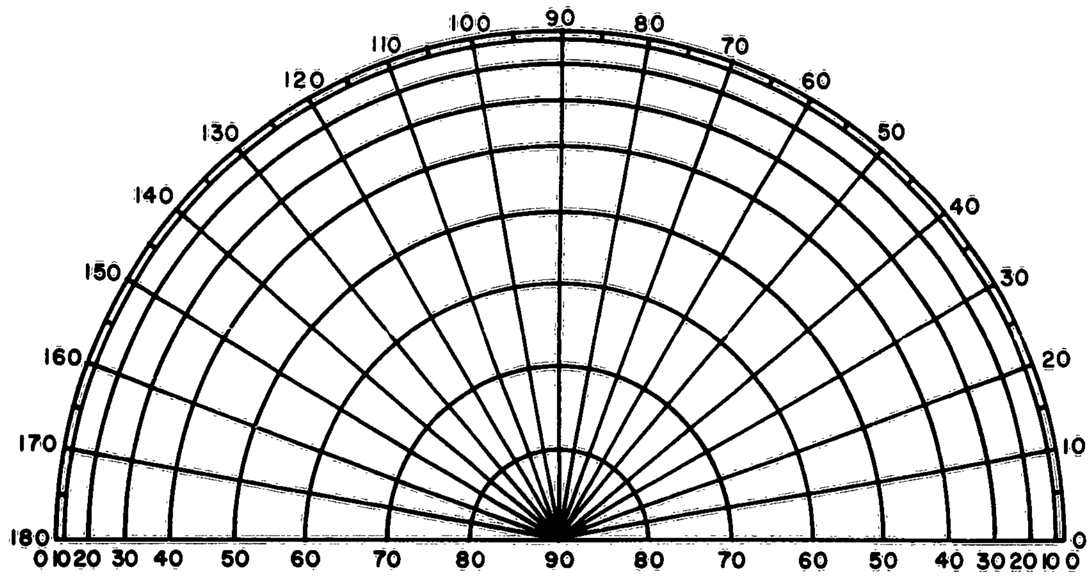

90
100
80
110
70
120
60
130
50
140
40
150
30
160
20
170
10
180
0
0 10 20 30 40 50 60 70 80 90 80 70 60 50 40 30 20 10 0

FIG. 19

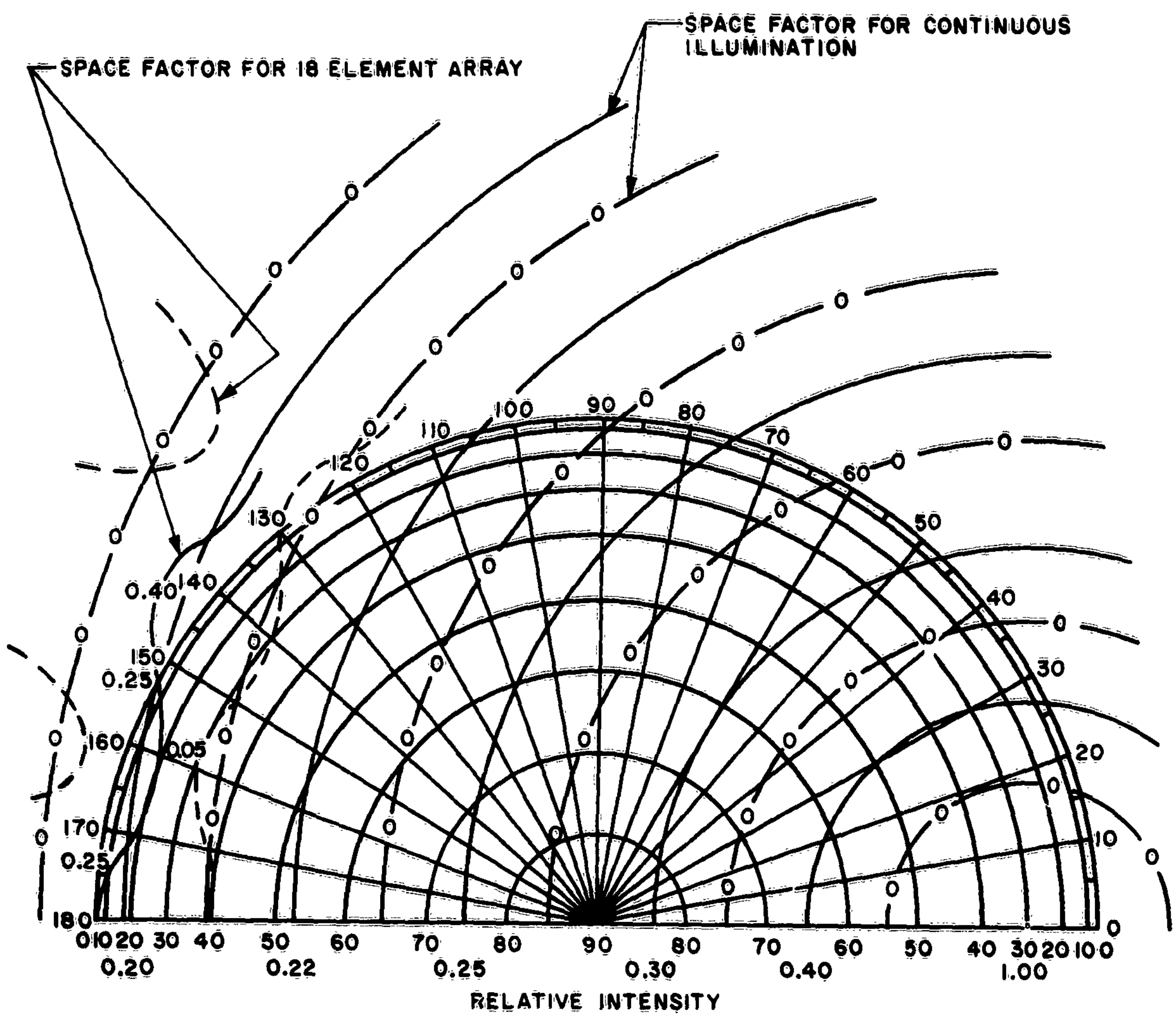

SPACE FACTOR FOR 18 ELEMENT ARRAY
SPACE FACTOR FOR CONTINUOUS ILLUMINATION
RELATIVE INTENSITY
0.20
0.22
0.25
0.30
0.40
1.00

FIG. 20

The performance of the antenna array as a function of frequency is found by using the computed pattern with graph papers in which the raddi of the circles are $\frac{f}{f_o}$ cos h. This follows from the general property of the patterns in transform space; the ratio of the radiation sphere to any reference dimension of the spectrum (for a fixed illumination over the aperture) is proportional to the frequency. The position of the center of the radiation circle is a function of frequency that depends on the particular phasing networks which are used. The center of the circle may follow any path (as a function of frequency) which is consistent with physically realizable phase-shift networks.

PART II

SUMMARY OF FORMULAS

INTRODUCTION

This section of the report summarizes the formulas which are used in the analysis and synthesis of the three broad classes of arrays. These are

(1) Perimeter arrays. In this class are

(a) Linear arrays, or straight line arrays. (Figure 21)

(b) Circular line arrays. (Figure 22)

(c) Rectangular line arrays. (Figure 23)

(2) Area or surface arrays. In this class are

(a) Plane rectangular arrays. (Figure 24)

(b) Plane circular area arrays. (Figure 25)

(c) Annular area arrays. (Figure 26)

(d) Cubical surface arrays. (Figure 27)

(e) Cylindrical surface arrays. (Figure 28)

Types (a) and (b) are basic and are possibly the most important of all. The formulas used in (c) are a special case of those used in (b). (d) and (e) are derived from lc and lb by extension.

(3) Volume arrays. In this class are

(a) Cubical volume arrays. (Figure 29)

(b) Cylindrical volume arrays. (Figure 30)

The formulas presented here provide a direct relationship between the illumination of the aperture and the space factor of the array. These formulas represent the aperture illumination in terms of a suitable Fourier series. Thus each component of the illumination is related to a particular component of the pattern in

LINEAR ARRAY

FIG. 21

CIRCULAR LINE ARRAY

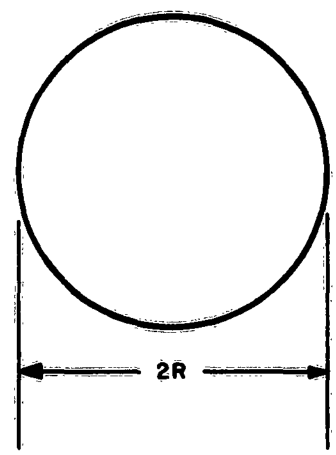


FIG. 22

RECTANGULAR LINE ARRAY

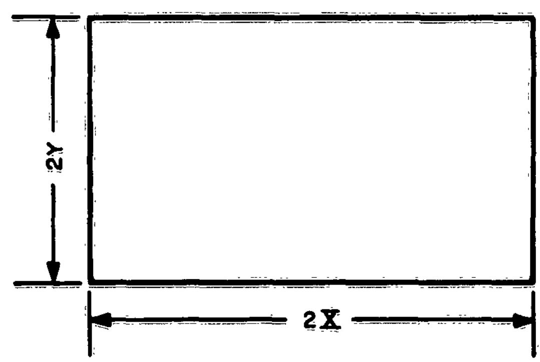


FIG. 23

PLANE RECTANGULAR ARRAY

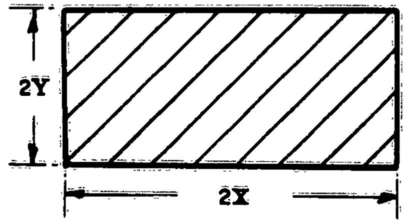


FIG. 24

PLANE CIRCULAR AREA ARRAY

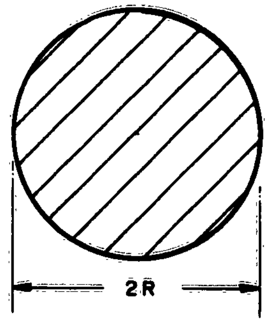


FIG. 25

ANNULAR RING ARRAY

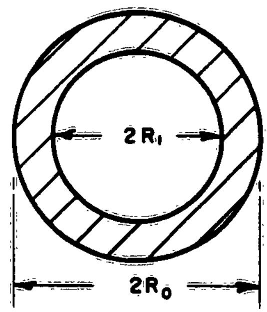


FIG. 26

CUBICAL SURFACE ARRAY

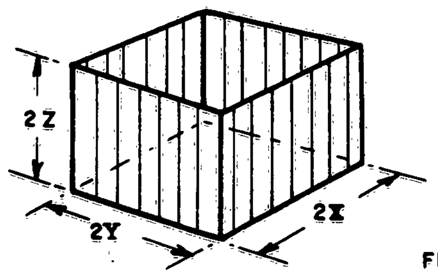


FIG. 27

CYLINDRICAL SURFACE ARRAY

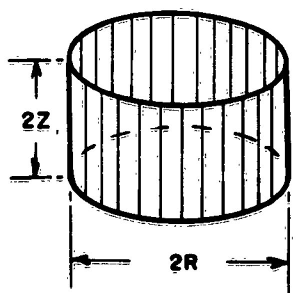


FIG. 28

RECTANGULAR

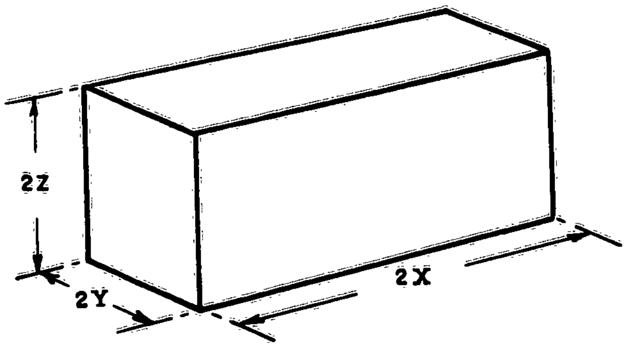


FIG. 29

CYLINDRICAL

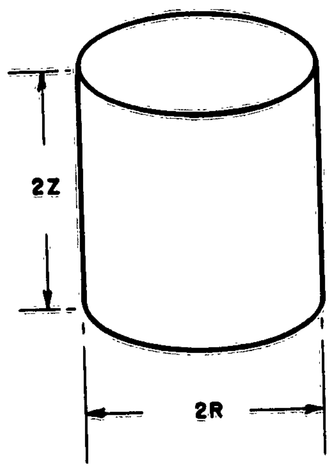


FIG. 30

transform space and the complete pattern is built up of the component building blocks. In all cases it is assumed that the voltages induced in all the antennas of the array are brought to some common point such as the origin by transmission lines of identical delay time and then added with suitable modifications of amplitude and phase which are defined by the aperture illumination and the initial phase adjustment. Reciprocal arrangements are made for transmitting arrays. An aperture illumination of unity is defined for the case where all of the voltages are added in phase and with equal amplitude <u>for a single direction of travel</u> of a plane wave, specified by $\alpha = \alpha_0$ and $h = h_0$.

The patterns are described in terms of the spectrum parameters, u, v, and w or ρ, θ and w. The spectrum parameters are trigonometrical and hence repetitive functions of azimuth and elevation.

THE GENERAL FORM OF EQUATION

The formulas which are presented here are all based on a set of basic relationships which are presented below. The phase and amplitude corresponding to any particular antenna compared to an antenna at the origin is expressible in these forms:

$$I = I(x,y,z)\,\epsilon^{j(ux+vy+wz)}$$

where

$$u = \cos\alpha \cos h - \cos\alpha_0 \cos h_0$$
$$v = \sin\alpha \cos h - \sin\alpha_0 \cos h_0$$
$$w = \sin h - \sin h_0$$

or $$I = I(r,\rho,z)\,\epsilon^{j(r\rho\cos(\beta-\theta) + wz)}$$

where $$\rho\cos(\beta-\theta) = \cos(\alpha-\beta)\cos h - \cos(\alpha_0-\beta)\cos h_0$$

From figure 15 it can be seen that

$$x = r \cos \beta \qquad u = \rho \cos \theta$$
$$y = r \sin \beta \qquad v = \rho \sin \theta$$

The general form is obtained by summing or, therefore, in the limit by integrating these terms in I over the entire region which contains the antennas. It is convenient to normalize the results by dividing the integral by a number equal to the extent of the region. This extent is a length or an area or a volume in any particular case. The general form of equation is therefore

$$S = \frac{1}{\text{extent}} \int_{\text{region}} I\,(\text{region})\ \epsilon^{j\phi\,(\text{region})}\ d\,(\text{extent})$$

This formula takes the forms shown in Table I for the basic types of arrays. The first column lists the types of array. The second column lists the basic formulas derived from the general form. The third column shows, for those types for which the results are useful, the inverse transforms by means of which I may be found from S, by direct application of Fourier's Integral Theorem.

Two questions might be raised from consideration of Table I. These concern

(1) The limitations upon the spectrum which will insure that the required aperture illumination will not differ from zero in the region outside the desired aperture and

(2) The inverse transforms for circular arrays.

These are answered below.

FOURIER EXPANSIONS OF THE ILLUMINATION AND OF THE SPECTRUM.

The basis for both the analysis and synthesis methods which are described in this report is the expansion of the aperture illumination into a suitable Fourier

From figure 15 it can be seen that

$$x = r \cos \beta \qquad u = \rho \cos \theta$$
$$y = r \sin \beta \qquad v = \rho \sin \theta$$

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(2) The inverse transforms for circular arrays.

These are answered below.

FOURIER EXPANSIONS OF THE ILLUMINATION AND OF THE SPECTRUM.

The basis for both the analysis and synthesis methods which are described in this report is the expansion of the aperture illumination into a suitable Fourier

series and the determination by means of the equations given in Table I of a suitable expansion of the spectrum in terms of a particular Fourier eigenfunction expansion. These eigenfunctions explicitly contain the dimensions of the aperture. Any spectrum built up out of any combination of any single set of eigenfunctions can be realized by an array contained wholly within the aperture, the size of which appears in the eigenfunction of the spectrum, and the form of which is determined by the form of the eigenfunction.

Tables II, III and IV show typical terms in the illumination of the several types of array considered here and the corresponding terms in the spectrum which are obtained by substituting the illumination function into the particular basic formula of Table I which is applicable to the type of array considered. The derivation of these formulas is discussed in Appendix I. The derivation of the formulas used for arrays of discrete antennas is given in Appendix II.

MATHEMATICS OF CIRCULAR ARRAYS.

In order to perform the integration in the circular case it is necessary to use the following relationship.

$$\epsilon^{j\rho r\cos(\beta-\theta)} = J_0(\rho R) + 2\sum_{p=1}^{\infty} (j)^p J_p(\rho R)\cos p(\beta-\theta)$$

Thus, for example, for a circular line array, Bessel functions are introduced.

In order to evaluate the circular area array, a double integration, with respect to both r and β is necessary. This may be done by integrating first with respect to β, and then integrating the Bessel function forms in ρR. There are only a few known indefinite integrals of Bessel functions; these were used to get the forms which are shown in Table III. Case 2 is a completely general form, in that any illumination can theoretically be expanded in a series known as a Dini Series, which is a form of Fourier-Bessel Series as follows:

$$I_K(\beta) = \mathcal{B}_K(\beta) + \sum_{M=1}^{\infty} I_{KM}\, J_K\left(\lambda_{KM}\frac{r}{R}\right)$$

In this form R is the largest radius of the aperture, the λ_{KM} are the roots of an auxiliary equation, and $\mathcal{B}_K(\beta)$ is an added term which is sometimes needed for this type of series. This form of expansion permits a solution for the spectrum in a form such as is shown in Table III, case 2. Alternative forms for case 2 of Table III and the derivation are given in the Appendix A.

INVERSE TRANSFORMS.

Fourier's Integral Theorem, which defines a pair of reciprocal transforms is most commonly known in the Campell-Foster form.

$$F(f) = \int_{-\infty}^{\infty} G(g)\, \epsilon^{-j2\pi fg}\, dg$$

$$G(g) = \int_{-\infty}^{\infty} F(f)\, \epsilon^{j2\pi fg}\, df$$

When F(f), and G(g) are real, this defines the relationship between two functions which can be represented by curves above an axis such as were shown in Figure 5. In general one or both functions may be complex, and a plane called the p-plane, where $p = j2\pi f$, is used; complex frequencies have been found a useful aid to computation.

In the multidimensional case the concept of complex frequencies is more complicated; however, with respect to each successive integration the tools of contour integration may be applied in mathematical extensions of the theory.

The inverse "rectangular" transforms always involve the same number of dimensions on both sides of the equations. This is not true, however, of the circular transforms. For example, a circular area array has the transform pair

$$\left\{ \begin{array}{l} \frac{1}{\pi R^2} \int_{-\pi}^{\pi} \int_{0}^{R} I(r,\beta)\, \epsilon^{j\rho r \cos(\beta-\theta)}\, r\,dr\,d\beta \\ \frac{R^2}{4\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} S(\rho,\theta)\, \epsilon^{-j\rho r \cos(\beta-\theta)}\, \rho\,d\rho\,d\theta \end{array} \right.$$

This has two dimensions in both cases, but the circular line array also requires a function of both ρ and θ for the inverse transform, and the solution only exists on the desired radius when the function $S(\rho, \theta)$ has the constraints indicated by the form of Table II, case 2. For this reason the inverse transform for this type was omitted from Table I.

The circular area case takes a special form when the formulas are written as follows:

$$I(r\theta) = \sum_{K=-\infty}^{\infty} I_K(r)\,\epsilon^{jK\theta}$$

and

$$S(\rho,\theta) = \sum_{K=-\infty}^{\infty} (j)^K \epsilon^{jK\theta} \phi_K(\rho)$$

where

$$\begin{cases} I_K(r) = \int_0^{\infty} \phi_K(\rho)\left[\rho J_K(\rho r)\right] d\rho \\ \phi_K(\rho) = \int_0^{R \text{ or } \infty} I_K(r)\left[r J_K(\rho r)\right] dr \end{cases}$$

These are the Bessel Transforms and are similar to the Fourier transforms; the Bessel Transforms are completely symmetrical and have a kernel of the form $t J_K(\omega t)$ instead of $\epsilon^{\pm j\omega t}$ as in the case of Fourier Transforms; the range of integration for the Bessel transform is from zero to infinity since it represents radial integration in a plane, and all radii are taken as positive.

TABLE I

BASIC FORMULAE FOR S (u, v)

I Type of Array	II Basic Formula	III Inverse Transform
Circular Line	$\frac{1}{2\pi R}\int_{-\pi}^{\pi} I(\beta)\epsilon^{jr\rho\cos(\beta-\theta)}d\beta$	
Straight Line	$\frac{1}{2X}\int_{-X}^{X} I(x)\epsilon^{jux}dx$	$\frac{X}{\pi}\int_{-\infty}^{\infty} S(u)\epsilon^{-jux}du$
Circular Plane Area Array	$\frac{1}{\pi R^2}\int_{-\pi}^{\pi}\int_{0}^{R} I(r,\beta)\epsilon^{jr\rho\cos(\beta-\theta)}r\,dr\,d\beta$	$\frac{R^2}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty} S(\rho,\theta)\epsilon^{-jr\rho\cos(\beta-\theta)}\rho\,d\rho\,d\theta$
Rectangular Plane Area	$\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y} I(x,y)\epsilon^{j(ux+vy)}dx\,dy$	$\frac{XY}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(u,v)\epsilon^{-j(ux+vy)}du\,dv$

TABLE I (Cont'd)

BASIC FORMULAE FOR S(u, v)

I Type of Array	II Basic Formula	III Inverse Transform
Rectangular Volume	$\frac{1}{8XYZ}\int_{-X}^{X}\int_{-Y}^{Y}\int_{-Z}^{Z} I(x,y,z)\epsilon^{j(ux+vy+wz)}\,dx\,dy\,dz$	$\frac{XYZ}{\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(u,v,w)\epsilon^{-j(ux+vy+wz)}\,du\,dv\,dw$
Cylindrical Volume	$\frac{1}{2Z\pi R^2}\int_{-\pi}^{\pi}\int_{0}^{R}\int_{-Z}^{Z} I(r,\beta,z)\epsilon^{j(r\rho\cos(\beta-\theta)+wz)}\,r\,dr\,d\beta\,dz$	$\frac{ZR^2}{4\pi^2}\int_{-\pi}^{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} S(\rho,\theta,z)\epsilon^{-j(r\rho\cos(\beta-\theta)+wz)}\,\rho\,d\rho\,d\theta\,dw$
General Form	$\frac{1}{extent}\int_{region} I(region)\epsilon^{j\phi(region)}\,d(extent)$	

TABLE II

PERIMETER ARRAYS

Type of Array	Illumination	Typical Terms of Spectrum
1. Linear	$I_K^{\theta} \epsilon^{jK\pi \frac{x}{X}}$	$I_K \frac{\sin(uX + K\pi)}{(uX + K\pi)}$
2. Circular	$I_K \epsilon^{jK\beta}$	$\frac{1}{R} I_K (j)^K J_K(\rho R) \epsilon^{jK\theta}$
3. Rectangular	$I_{K1} \epsilon^{jK\pi \frac{x}{X}} + I_{K2} \epsilon^{jK\pi \frac{y}{Y}}$	$2 I_{K1} \cos vY \left(\frac{\sin(uX + K\pi)}{uX + K\pi} \right) + 2 I_{K2} \cos uX \left(\frac{\sin(vY + K\pi)}{vY + K\pi} \right)$

TABLE III

AREA ARRAYS

Type of Array	Typical Terms of Illumination	Typical Terms of Spectrum
1. Rectangular	$I_{KM}\,\epsilon^{j\left(K\pi\frac{x}{X}+M\pi\frac{y}{Y}\right)}$	$I_{KM}\ \frac{\sin(uX+K\pi)}{uX+K\pi} - \frac{\sin(vY+M\pi)}{vY+M\pi}$
2. Circular (General)	$I_{KM}\,J_K\left(\lambda_{KM}\frac{r}{R}\right)\epsilon^{jK\beta}$ $\lambda_{KM}{}^{-M}\left[\lambda_{KM}J'_K(\lambda_{KM})+HJ_K(\lambda_{KM})\right]=0$	$2I_{KM}(j)^K\epsilon^{jK\theta}J_K(\lambda_{KM})\frac{\left(H\frac{\lambda_{KM}}{\rho R}\pm K\right)J_K(\rho R)\mp\left(J_{K\pm1}(\rho R)\right)\rho R}{\lambda_{KM}^2-(\rho R)^2}$
3. Circular (Limited)	$I_{KM}\left(\frac{r}{R}\right)^K\left(1-\frac{r^2}{R^2}\right)^M\epsilon^{jK\beta}$ $M>-1$	$I_{KM}(j)^K\epsilon^{jK\theta}2^{M+1}\,\Gamma(M+1)\ \frac{J_{K+M+1}(\rho R)}{(\rho R)^{M+1}}$
4. Circular Ring Type	$I_K\left(\frac{r}{R_o}\right)^{\pm K}\epsilon^{jK\beta}$ $R_o \geqslant r \geqslant R_1$	$2I_K(j)^K\cdot\epsilon^{jK\theta}\left[\frac{J_{1\pm K}(\rho R_o)}{\rho R_o}-\left(\frac{R_1}{R_o}\right)^{2\pm K}\frac{J_{1\pm K}(\rho R_1)}{\rho R_1}\right]$
5. 4-faced Cubical Surface	$I_{KL}\,\epsilon^{j\left(K\pi\frac{x}{X}+L\pi\frac{z}{Z}\right)}+I_{ML}\,\epsilon^{j\left(M\pi\frac{y}{Y}+L\pi\frac{z}{Z}\right)}$	$2I_{KL}\cos vY\ \frac{\sin(uX+K\pi)\sin(wZ+L\pi)}{(uX+K\pi)(wZ+L\pi)}$ $+2I_{ML}\cos uX\ \frac{\sin(vY+M\pi)\sin(wZ+L\pi)}{(vY+M\pi)(wZ+L\pi)}$
6. Vertical Cylindrical Surface	$I_{KM}\,\epsilon^{j\left(K\beta+M\pi\frac{z}{Z}\right)}$	$\frac{1}{R}I_{KM}(j)^K J_K(\rho R)\,\epsilon^{jK\theta}\ \frac{\sin(wZ+M\pi)}{(wZ+M\pi)}$

TABLE IV

VOLUME ARRAYS

Typical Terms of

	Illumination	Spectrum
1. Rectangular Volume	$I_{KML}\,\epsilon^{j\left(K\pi\frac{x}{X}+M\pi\frac{y}{Y}+L\pi\frac{z}{Z}\right)}$	$I_{KML}\,\dfrac{\sin(uX+K\pi)\,\sin(vY+M\pi)\,\sin(wZ+L\pi)}{(uX+K\pi)\,(vY+M\pi)\,(wZ+L\pi)}$
2. Vertical Cylindrical Surface	$\epsilon^{j\left(L\pi\frac{z}{Z}\right)}\times\begin{Bmatrix}\text{III}-2\\ \text{III}-3\\ \text{III}-4\end{Bmatrix}$	$\dfrac{\sin(wZ+L\pi)}{(wZ+L\pi)}\times\begin{Bmatrix}\text{III}-2\\ \text{III}-3\\ \text{III}-4\end{Bmatrix}$

PART III
SYNTHESIS OF ARRAYS

INTRODUCTION

This section of the report presents details of how arrays may be synthesized by the eigenfunction method and discusses some of the problems which are involved when the synthesis method is applied. A numerical example of the synthesis of a rectangular array is given.

In conclusion, the properties of circular arrays are considered in detail although numerical examples for circular arrays are beyond the scope of this report.

THE STEPS IN THE SYNTHESIS METHOD

The problems involved in the synthesis of an array become apparent when the steps in the synthesis method are considered in detail. These are as follows:

(1) The design is undertaken to obtain a particular form of radiation pattern. This is defined in terms of the description of the required size and shape of the major lobe and allowable size and shape of the minor lobes. Often there are other limitations which may restrict the size, shape and complexity of the array. The first step is to clearly define what is desired.

(2) A type of array (such as for example a plane rectangular array or a vertical cylinder array or some other type of array) must be chosen tentatively and the design carried out with this type of array to find what degree of complication, what size array, and how many elements are required in the array in order to meet the specified conditions. Since there are several types of arrays adequately general in physical characteristics to produce any arbitrary pattern, there may ultimately be a choice among several designs on the basis of economy, simplicity and practicality for the particular application.

(3) A tentative choice must be made of the basic elements of the array taking account of any ground plane or other reflecting surface which can modify the operation of the array in actual use. The design of the array therefore should include the effect of ground planes. Under certain conditions the orientation of the elements in the array with respect to the array gives a useful additional variable which can be controlled to help obtain the desired overall result economically.

(4) At this point the required space factor may be defined and the space factor can be synthesized from the eigenfunctions for the particular type of array being considered. Thus the coefficients in the generalized Fourier representation of the aperture illumination may be determined.

(5) The next step is to find the minimum number of antennas which will eliminate spurious back lobes. At this point it is sometimes desirable to reconsider whether it may be possible to reduce the necessary number of antennas by modifying the type or orientation of the basic elements in the array.

(6) It is sometimes desirable to go through the entire design procedure for more than one type of array and to compare the end results. However, if it is merely desired to find any method which gives the desired results or if there are specific reasons why a particular form of array must be used the design is completed with the preceding step.

DIPOLE ARRAYS

The overall radiation pattern of an array is the product of the space factor of the array and the radiation pattern of the elements. Figure 31 shows plane dipole arrays with three different orientations of the dipoles with respect to the planes of the arrays. In each case there is shown also how the pattern of a single dipole may be represented in the radiation circle of the transform plane. The

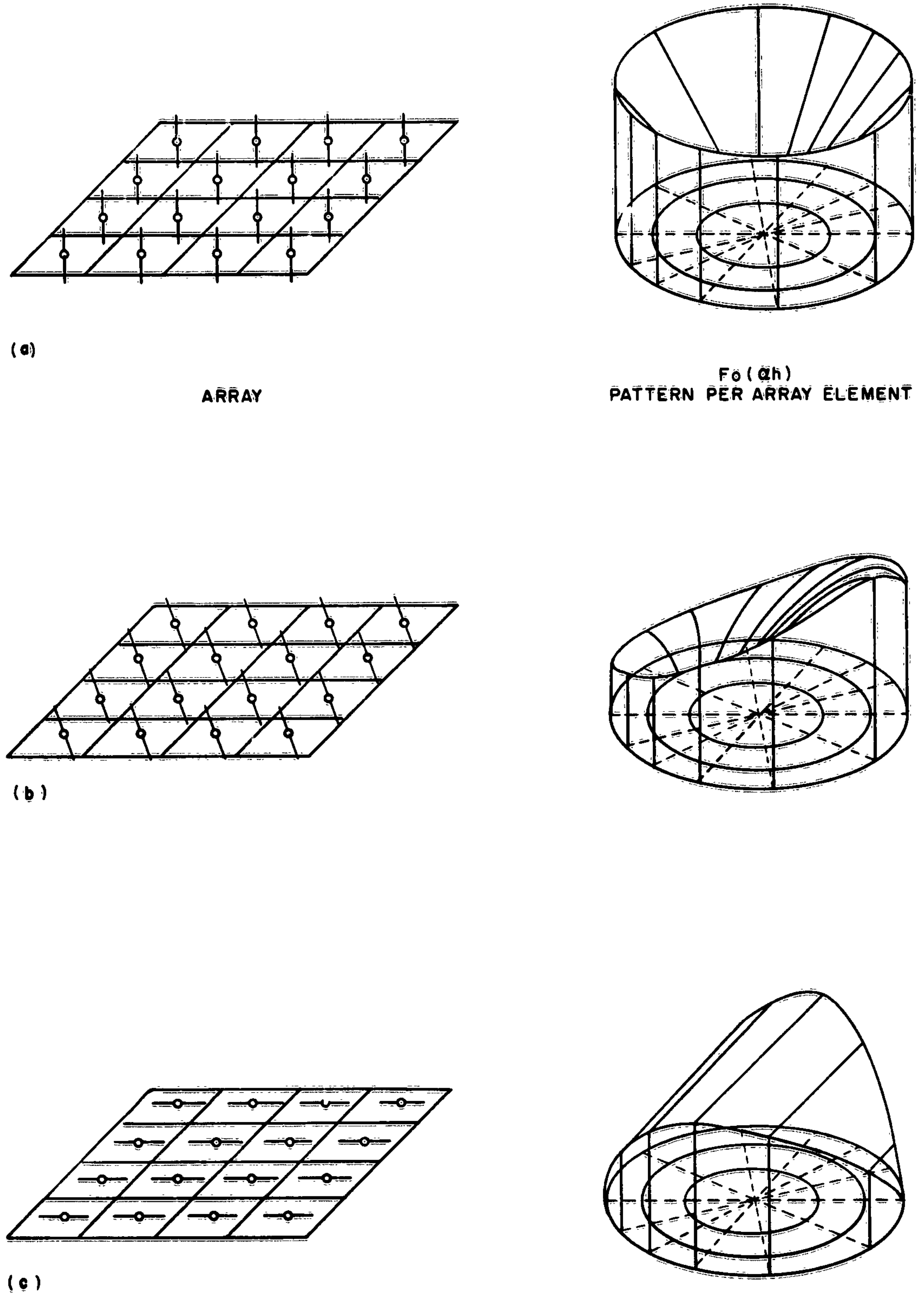


FIG.31

circle has coordinates here of the type shown earlier for horizontal plane arrays. For each azimuth and elevation there is indicated a relative radiation intensity from the dipole. The resulting surface defines a function of azimuth and elevation which multiplies the space factor at points in the transform plane which correspond to points of radiation space. This is the F_0 function which was mentioned in Part I.

The concept of the space factor may be readily extended to what may be called "arrays of arrays," for which the overall pattern is the product of the pattern per element, the space factor per basic array group and the space factor for the set of groups. The procedure of developing an array into an array of arrays has been called convolution, and an alternative approach which gives these results is to apply the familiar convolution or folding integral of Fourier integral analysis. See for example "Fourier Integrals for Practical Applications," G. A. Campbell and R. M. Foster, pairs #202 and #203. The convolution process can be used in either direction. For example, the results obtained for arrays of discrete antennas give, for the product of the illuminations, the convolution of the patterns.

PULSE SEQUENCES

There are many types of pulse shapes which can be built from the $\frac{\sin x}{x}$ type of eigenfunction. One simple and useful group of these is shown in Figure 32. This group is called the flat-field sequence and is one group which gives very small side lobes. With some slight modifications of these, the first side lobe can easily be reduced to zero amplitude. However, in a practical case tolerance problems would make such close design futile.

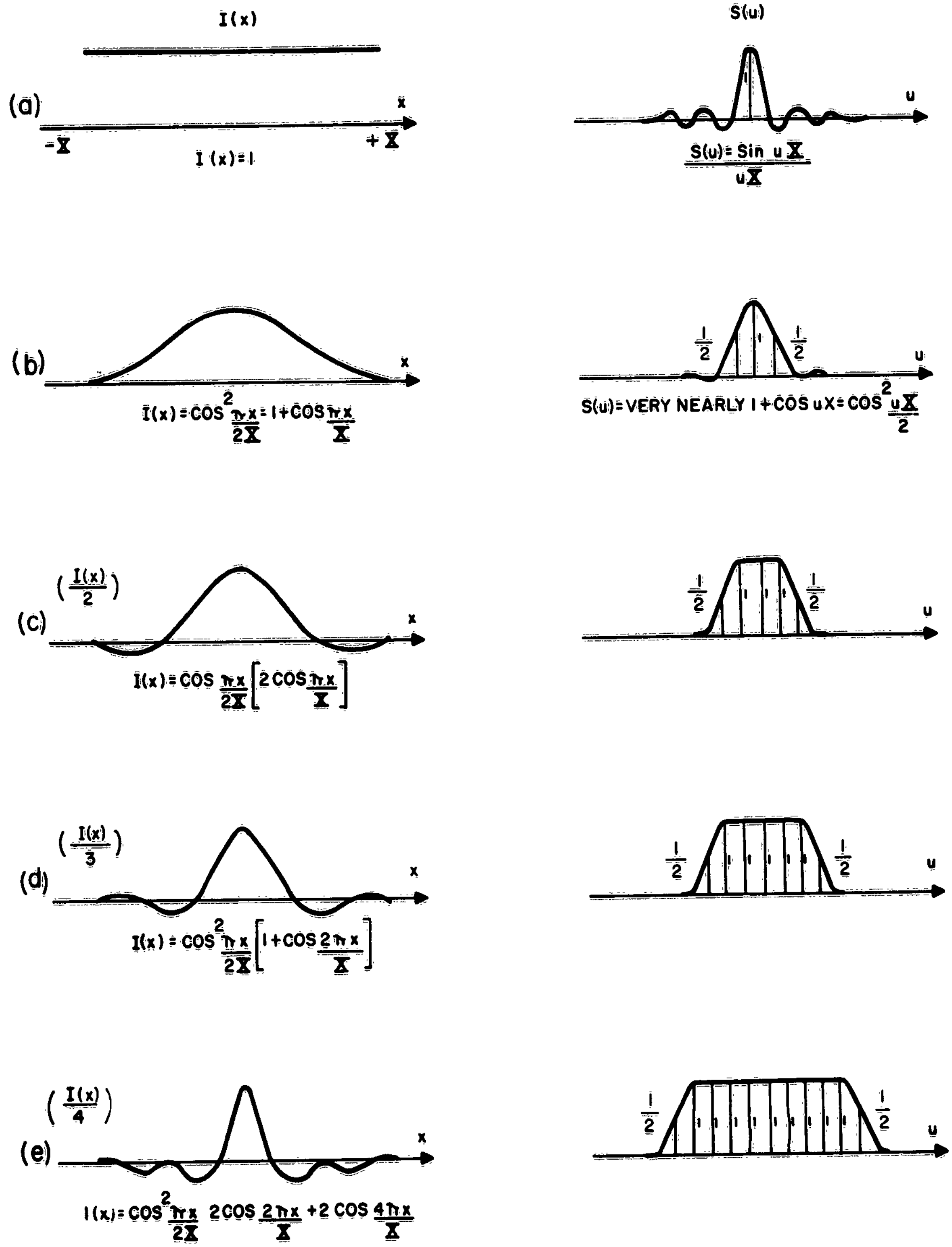

I(x)
S(u)
(a)
x
-X
+X
I(x)=1
u
S(u)= Sin uX / uX
(b)
I(x)=COS² πx/2X = 1+COS πx/X
1/2
S(u)=VERY NEARLY 1+COS uX=COS² uX/2
(c)
(I(x)/2)
I(x)=COS πx/2X [2COS πx/X]
(d)
(I(x)/3)
I(x)=COS² πx/2X [1+COS 2πx/X]
(e)
(I(x)/4)
I(x)=COS² πx/2X 2COS 2πx/X +2 COS 4πx/X

FIG.32

Another very useful type is shown in Figure 33b. Figure 33 compares the $\frac{\sin x}{x}$, cosine and cosine-squared types of illumination. Use of the illumination specified for Figure 33b instead of that specified for Figure 33c in an array of the type having a single major lobe permits a saving of 36% in the number of antennas required for the same 3 db and 6 db beamwidths.

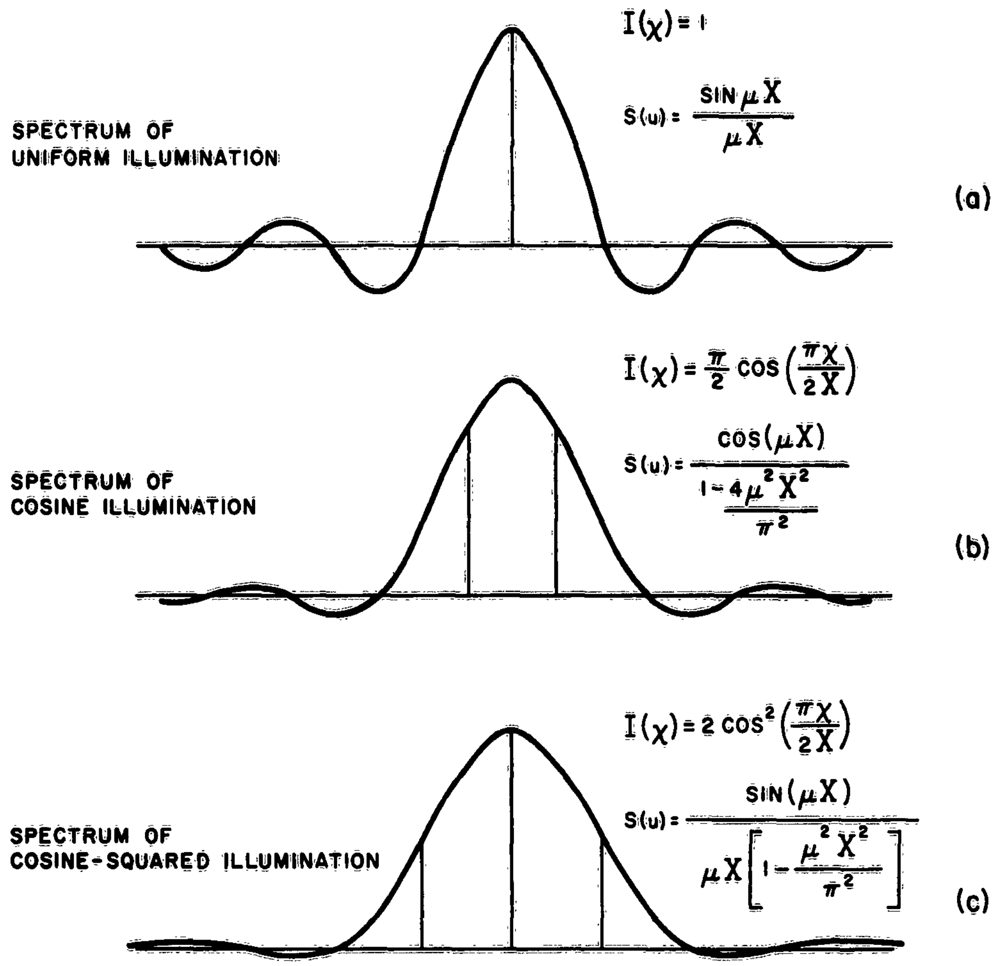
Patterns which are built up of eigenfunctions for which the independent points are all within the radiation circle generally permit the exchange of beamwidth for side-lobe level. However, some patterns which use independent points outside the region of the transform plane corresponding to the radiation circle can exchange number of harmonics (or of antennas) for either or both of the above. The resulting patterns are called superdirective. A basic superdirective type of pattern for the linear array is shown in Figure 34, along with the required illumination for it. The same principles are applicable to the other types of arrays.

EXAMPLES OF SYNTHESIS PLANE RECTANGULAR ARRAYS:

Two examples will be given: the first a qualitative example to outline in detail what the steps are in the synthesis process; and the second a numerical example.

Qualitative Example, Vertical Wedge Beam:

Suppose it were desired to synthesize an array subject to the following limitations: only vertical antennas above a ground plane may be used; the side lobes shall be as small as is reasonably possible; and at every elevation the main lobe shall correspond to the same value of azimuth as for every other elevation. The design is based on a dipole array of the type shown in Figure 31a. The steps in

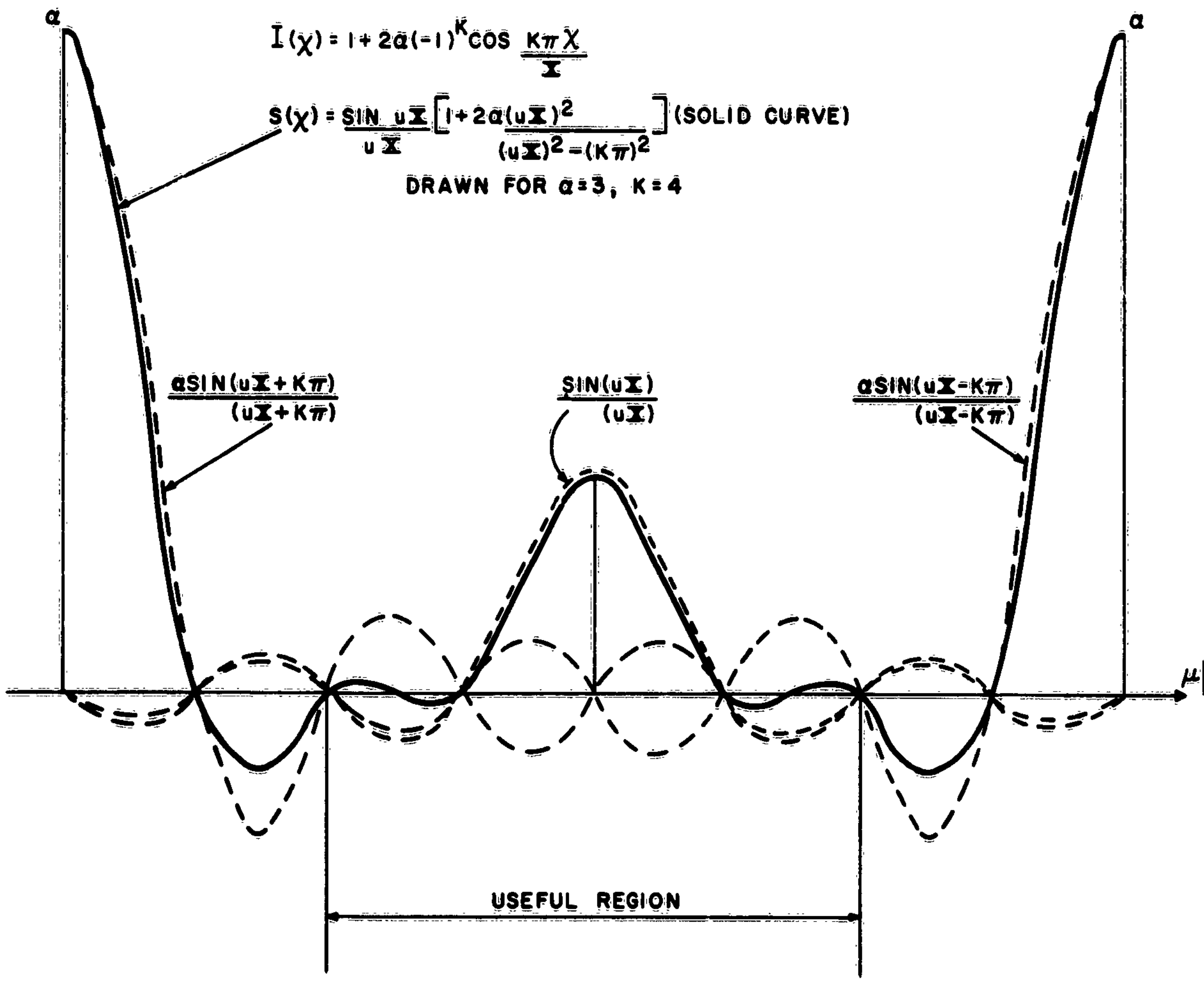


COMPARATIVE TABLE

RATIO OF TOTAL WIDTH TO SPACING BETWEEN INDEPENDENT POINTS, FOR

ILLUMINATION	1 db down	3 db down	6 db down	FIRST ZERO	db down at first side lobe
uniform	K = 0.6	0.9	1.2	2.0	13.5
cos	0.6	1.2	1.6	3.0	24
$\cos^2$	0.8	1.5	2.0	4.0	32

FIG. 33



a
$I(\chi) = 1 + 2a(-1)^K \cos \frac{K\pi\chi}{X}$
$S(\chi) = \frac{\sin uX}{uX}\left[1 + 2a\frac{(uX)^2}{(uX)^2-(K\pi)^2}\right]$ (SOLID CURVE)
DRAWN FOR a=3, K=4
a
$\frac{a\sin(uX+K\pi)}{(uX+K\pi)}$
$\frac{\sin(uX)}{(uX)}$
$\frac{a\sin(uX-K\pi)}{(uX-K\pi)}$
μ
USEFUL REGION

FIG. 34

the design procedure are indicated in Figure 35 as follows: The space factor of the array is first sketched on a design chart such as is shown in Figure 35a. The single lobe shown here is built up from the pulse sequence of Figures 32c and 33b. The dimensions of the aperture of the array are found from the desired beam width as shown in Figure 35a. This determines the locations of the independent points as is shown in Figure 35b. The last step, shown in Figure 35c consists of finding the minimum number of antennas in the array which will prevent the occurrence of undesirable side lobes due to the discreteness of the array. Since the pattern repeats in the u and v directions at distances corresponding to a number of independent points equal to the number of rows in the x and y directions, the number of antennas is equal to the number of independent points within the dotted rectangle shown in Figure 35c.

A Numerical Example:

As a numerical example suppose it were necessary to find the plane area and approximate minimum number of antennas per unit area required to produce an array having a beam with a single major lobe and very small minor lobes, and subject to the following conditions.

(1) The half-amplitude azimuthal beamwidth shall be about $\pm 18^{\circ}$ at zero elevation. This is 36° total.

(2) The main beam shall be at the same azimuth for all elevations.

(3) The half-amplitude vertical beamwidth shall be adequate for elevations up to at least 60°.

(4) The illuminations shall be as simple as possible.

(5) The array must consist of vertical antennas above a plane earth.

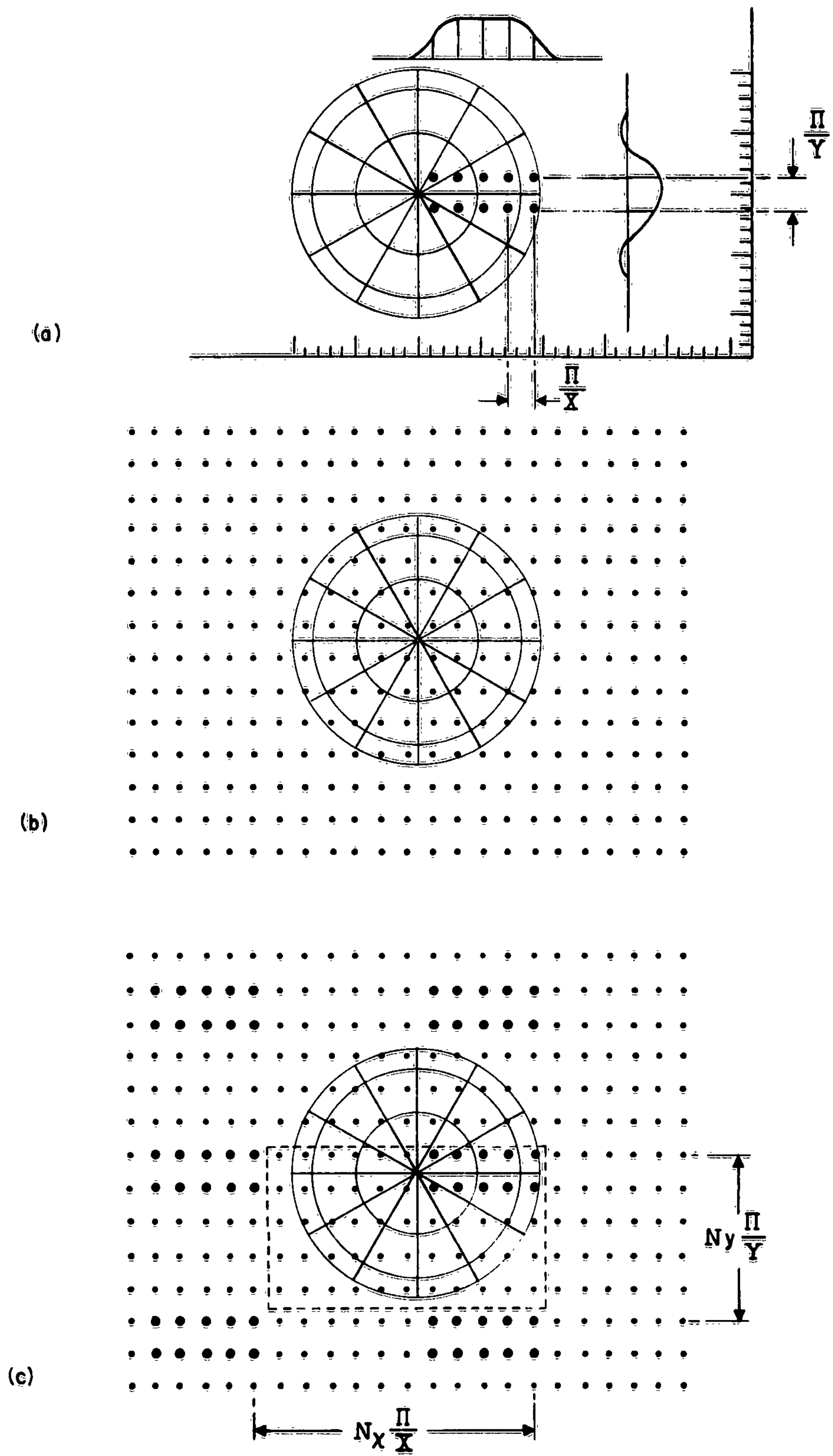


FIG.35

An illumination of the type specified for Figure 33b will be used in both x and y directions.

Let $u' = u + \cos h_o$.

For this type of pattern the half-amplitude points of the space factor occur at a separation $\Delta u' = 1.64 \frac{\pi}{x} \approx 1.6 \frac{\pi}{x}$. Let these points correspond to $\alpha = 0^o$, and $h = 0^o$ and $h = 78.5^o$ (see Figure 36). Then $\Delta u' = 0.8 = \cos 0^o - \cos 78.5^o = \frac{1.6\pi}{x}$ or the aperture $= 2X = 4\pi$.

The nominal beam center of the space factor is half way between these points, or at $u' = 0.2 + \frac{0.8}{2} = 0.6$.

For a half beamwidth of about 18^o let $\Delta\nu = 1.6 \frac{\pi}{Y} = 2(0.3) = 0.6$ or the aperture $= 2Y = 5\ 1/3\pi$.

The independent points in the u direction appear at $u' = 0.6 + \frac{\pi}{2x} + \frac{K\pi}{x}$. But $0.6 + \frac{\pi}{2x} = 0.6 + \frac{\pi}{4\pi} = 0.85$. The spacing between independent points is $\frac{\pi}{x}$ which is 0.5. Therefore the independent points occur at $u' = 1.35$, .85, .35, -.15, -.65, -1.15, -1.65, etc. These are shown in Figure 36. The value $u' = .85$ determines $h_o = 32^o$.

The first independent points in the v direction occur at $/\nu/ = \frac{.3}{1.6} = .188$. Since the spacing between independent points along the v axis is $\Delta\nu = \frac{\pi}{Y} = .375$ the independent points occur at $/\nu/ = .188$, .563, .938, 1.313 etc. These are shown in Figure 36.

In order to have enough antennas to eliminate side lobes but not more than this number, let the first zero of the repeated (discreteness) pattern along the u' axis occur at $u' = -1.15$. This is outside the radiation circle. The corresponding zero of the main pattern occurs at $u' = 1.35$. Therefore 5 rows are needed in the x direction.

An illumination of the type specified for Figure 33b will be used in both x and y directions.

Let $u' = u + \cos h_o$.

For this type of pattern the half-amplitude points of the space factor occur at a separation $\Delta u' = 1.64 \frac{\pi}{x} \approx 1.6 \frac{\pi}{x}$. Let these points correspond to $\alpha = 0^o$, and $h = 0^o$ and $h = 78.5^o$ (see Figure 36). Then $\Delta u' = 0.8 = \cos 0^o - \cos 78.5^o = \frac{1.6\pi}{x}$ or the aperture $= 2X = 4\pi$.

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The independent points in the u direction appear at $u' = 0.6 + \frac{\pi}{2x} + \frac{K\pi}{x}$.

But $0.6 + \frac{\pi}{2x} = 0.6 + \frac{\pi}{4\pi} = 0.85$. The spacing between independent points is $\frac{\pi}{x}$ which is 0.5. Therefore the independent points occur at $u' = 1.35$, .85, .35, -.15, -.65, -1.15, -1.65, etc. These are shown in Figure 36. The value $u' = .85$ determines $h_o = 32^o$.

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In order to have enough antennas to eliminate side lobes but not more than this number, let the first zero of the repeated (discreteness) pattern along the u' axis occur at $u' = -1.15$. This is outside the radiation circle. The corresponding zero of the main pattern occurs at $u' = 1.35$. Therefore 5 rows are needed in the x direction.

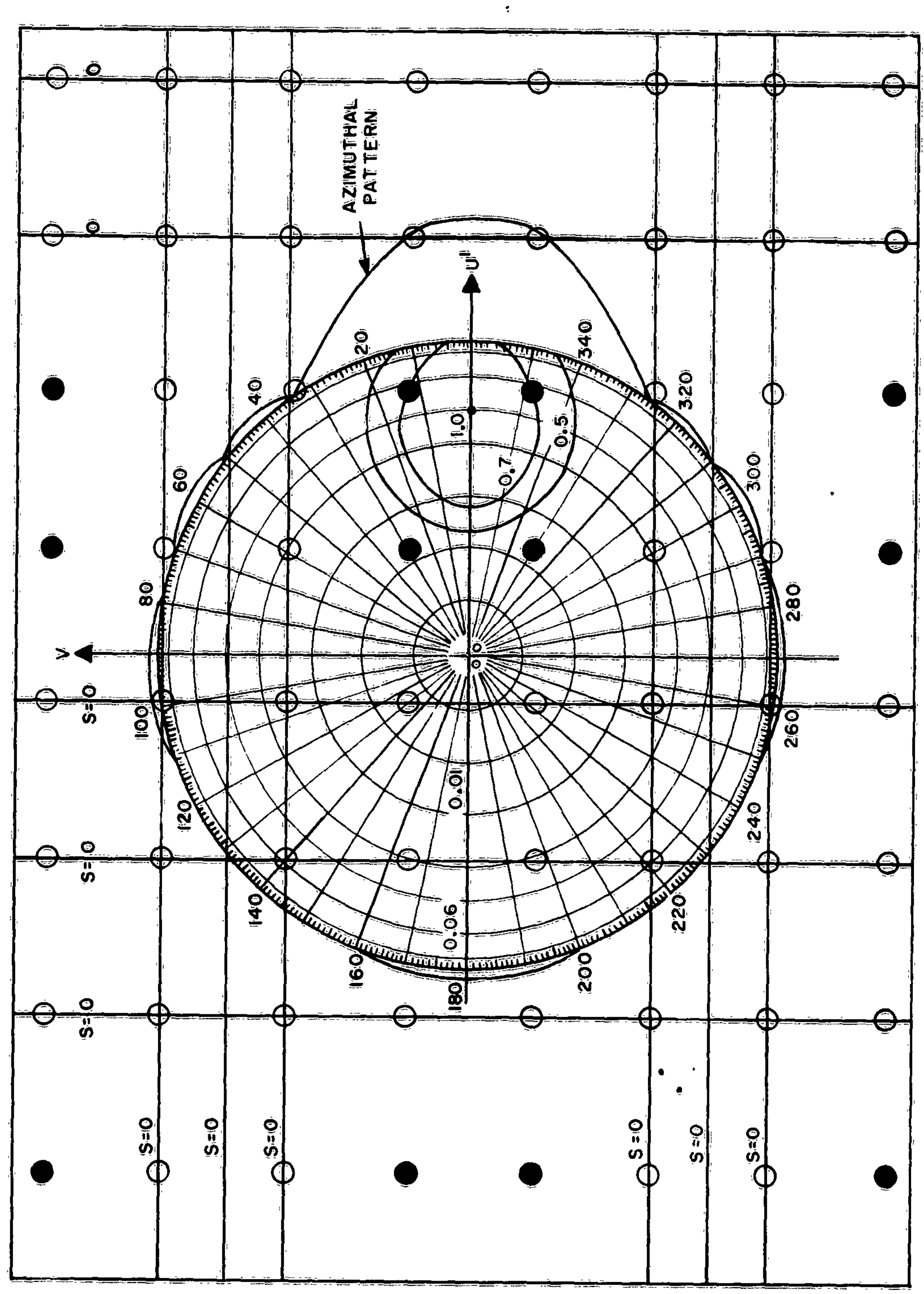

AZIMUTHAL PATTERN
U
V
S=0
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
1.0
0.7
0.5
0.06
0.01

FIG. 36

The first zero in the v direction occurs at v = -.563. Let the corresponding zero of the repeated (discreteness) pattern occur at v = .938; then 4 rows are needed in the y direction. The black dots in Figure 36 indicate the basic and repeated patterns. Then 4 x 5 = 20 antennas are required. The locations of the antennas are shown in Figure 37. The equation for the illumination is

$$I(x, y)\epsilon^{-jx\cos h_0} = \left[\cos\frac{\pi x}{2X}\cos\frac{\pi y}{2Y}\right]\epsilon^{-jx\cos h_0}$$

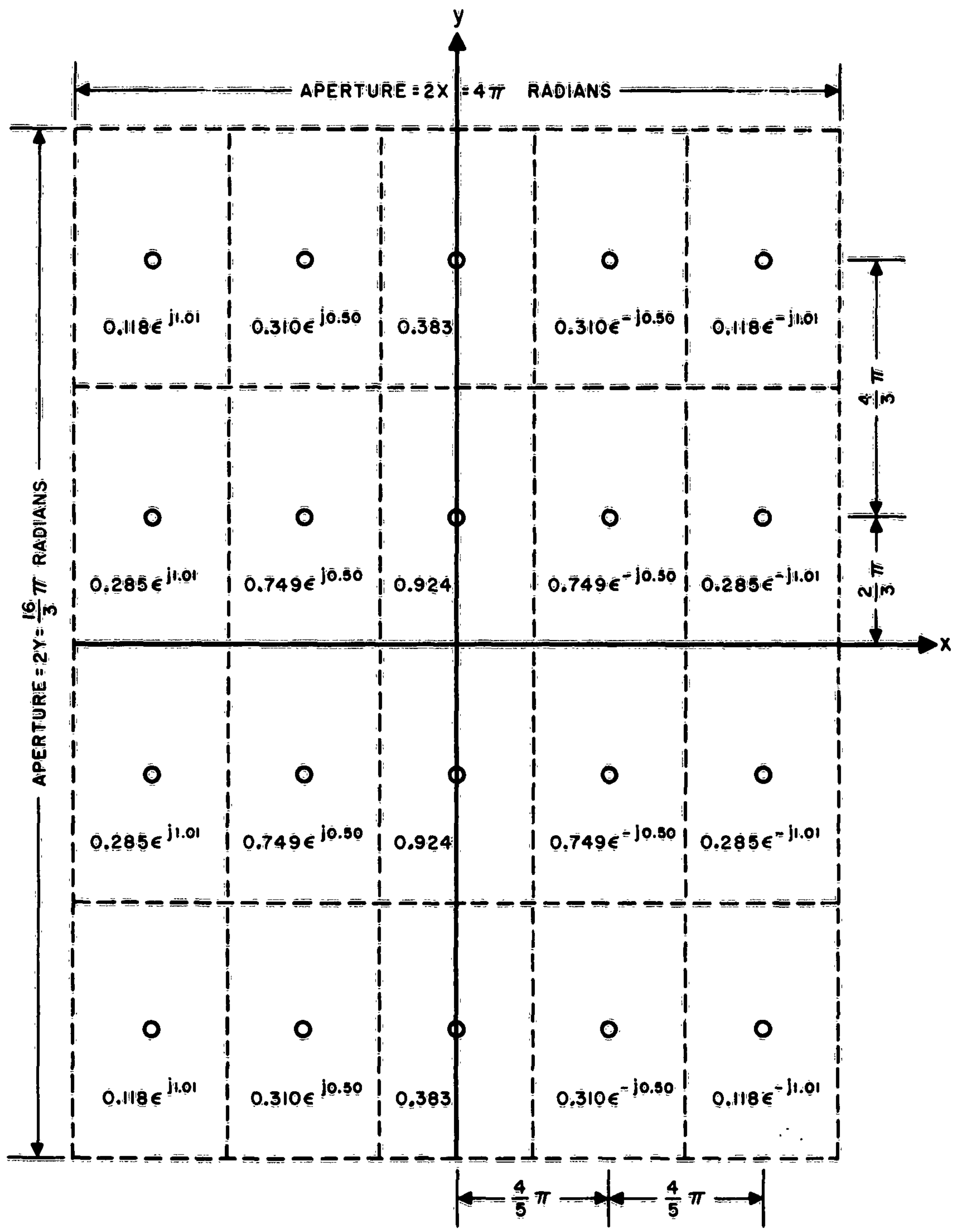
Figure 37 tabulates the relative amplitudes and phases of the illumination for the twenty antennas, including the phase adjustment which would make all of the contributions of the individual antennas add in phase if the aperture were uniformly illuminated.

Contours of the overall radiation pattern including the F_0 factor for the dipole of the type shown in Figure 31a are shown in Figure 36. Figure 36 also shows the horizontal azimuthal pattern, which is sketched around the unit circle.

Some of the physical characteristics of circular arrays will be briefly discussed below, but a numerical example is beyond the scope of this report.

THE REQUIRED NUMBER OF ANTENNAS IN AN ARRAY OF ARBITRARY SHAPE

The synthesis of arrays of antennas of circular or other shapes requires a knowledge of how many antennas are needed in order to prevent the appearance of spurious lobes due to discreteness effects. Physically the problem is simple, since for example any plane array can be considered to be contained within some convenient rectangle. In the circular case the rectangle becomes a square in which the illumination is different from zero only within a circular region. The



The circular dots give the locations of the antennas. The numbers define the illuminations of the antennas, as determined in the Numerical Example.

FIG. 37

effect of discreteness is to make the pattern repeat at such intervals as are indicated in Figure 35c. For example, computations on the circular array with reflector show that when the frequency is increased by 50%, and hence the ratio of radiation circle size to space factor pattern size is increased, side lobes appear in the region of about $\pm 80^{\circ}$ as would be expected from Figure 35c.

Thus, the (approximate) required density of antennas, that is, number of antennas per unit area in order to control side lobe effects, may be readily determined for any shape.

Sometimes an array may be considered most conveniently as the sum of two arrays and the overall pattern is then the sum (with due regard to phase angles) of the two patterns; in such a case the discreteness problem is solved for each component array separately.

The same concepts are applicable to volume arrays and to spectra in three dimensional transform space. Thus at this point several broad conclusions may be stated regardless of the shape of the aperture:

(1) The coarseness or basic ripple or grain size in transform space is determined by the effective size of the aperture in the coordinate direction corresponding to the transform coordinate along which the grain size is measured. The words effective size are used because in general patterns which have small amplitude near the edges have broader main lobes than apertures of the same size which are uniformly illuminated.

(2) A direct exchange of beamwidth for side lobe level may be made by using illuminations which are largest near the center and which fall to a small value near the edges of the array.

(3) There is a direct relationship between the density of antennas in any direction in array space and the distance between the repeated patterns in the corresponding direction in transform space. This is a physical phenomenon which is independent of the exact shape of the array.

SYNTHESIS OF CIRCULAR ARRAYS

The synthesis of circular arrays is complicated by the fact that the Bessel functions from which the patterns must be built up are not quite as easy to use as sinusoidal functions. However, if the various characteristics of the $\frac{\sin x}{x}$ functions which have been found useful are enumerated then these same characteristics may be found in certain of the forms obtained for circular arrays.

These useful properties are:

(1) It is possible to build basic patterns by use of the independent point method.

(2) It is simple with this particular form of eigenfunction to exchange beamwidth for side lobe level directly. There are known sets of pulse shapes which can be easily designed to produce small side lobes.

(3) The eigenfunctions are simple and of a familiar form.

The terms which appear in the spectrum column of items 2 and 3 of Table III will be briefly discussed below. It will be shown that Type III-2 possess the first property and that Type III-3 possess the second property. In both cases the physical characteristics of the functions, regardless of the values of the indices K and M, are simple enough so that a little study produces the desired familiarity.

The Radiation Function (Type III-2)

Consider the function

$$R_{KM}(\rho R)\epsilon^{jK\theta} = J_K(\lambda_{KM}) \frac{\left(H\frac{\lambda_{KM}}{\rho R} \pm K\right) J_K(\rho R) \mp \rho R\, J_{K\pm 1}(\rho R)}{(\lambda_{KM})^2 - (\rho R)^2}$$

where

$$\lambda_{KM}^{-M}\left(\lambda_{KM} J_K'(\lambda_{KM}) + H J_K(\lambda_{KM})\right) = 0$$

This results from expanding an arbitrary function of r in a Fourier-Bessel Series.

In order to find out how the function $R_{KM}(\rho R)$ behaves as the parameter H varies there are tabulated in Table V the forms which $R_{KM}(\rho R)$ takes when H takes on several representative values over the range which it can have. Each value of H corresponds to a specific ratio between the derivative of the radial function and the radial function itself which is associated with the K'th (exponential) harmonic of the aperture illumination.

Then

$$H = -\frac{1}{R} \frac{I_K'(r)}{I_K(r)}\bigg|_{r=R}$$

TABLE V

H	$R_{KM}(\rho R)$
∞	$-\frac{\lambda_{KM}}{\rho R} \frac{\lambda_{KM} J_K'(\lambda_{KM}) J_K(\rho R)}{\lambda^2_{KM} - (\rho R)^2}$
$\pm K$	$J_K(\lambda_{KM}) \frac{\rho R\, J_K'(\rho R) \pm K \frac{\lambda_{KM}}{\rho R} J_K(\rho R)}{\lambda^2_{KM} - (\rho R)^2}$
0	$J_K(\lambda_{KM}) \frac{\rho R\, J_K'(\rho R)}{\lambda_{KM}^2 - (\rho R)^2}$

All the Bessel Functions of the first kind, J_K (ρR) behave very much like sinusoidal functions which are attenuated in amplitude as $\frac{1}{\sqrt{\rho R}}$ for $\rho R > K$, and are substantially zero for $\rho R < K$ approximately (refer to Figure 16).

The numerator functions here behave much like Bessel Functions, and hence have almost regularly spaced zeros spaced by very nearly $\frac{\pi}{R}$. For large values of ρR the functions behave as

$$\frac{\pm K \; J_K(\rho R) \pm \rho R \, J_{K \pm 1}(\rho R)}{\lambda_{KM}^2 - (\rho R)^2}$$

For small values of ρR the functions behave as $\dfrac{\dfrac{J_K(\rho R)}{\rho R}}{\lambda^2_{KM} - (\rho R)^2}$

The number H determines the numerator function. The possible roots λ_{KM} which are the eigenvalues of the eigenfunctions used in the Fourier expansion of the spectrum are all roots of the function which appears in the numerator. The characteristics of the function can be seen from Figure 38 in which Figure 38a shows the performance of the numerator function, Figure 38b shows the performance of the denominator function and Figure 38c shows the composite function for the case where $H = \infty$. The composite function has the same zeros as the numerator function with the one exception, λ_{KM}. The composite function behaves very much like the numerator function until almost up to the value corresponding to the root, λ_{KM}. One zero is then removed just like with the $\frac{\sin x}{x}$ functions and for higher values of ρR, R_{KM} (ρR) is attenuated rapidly.

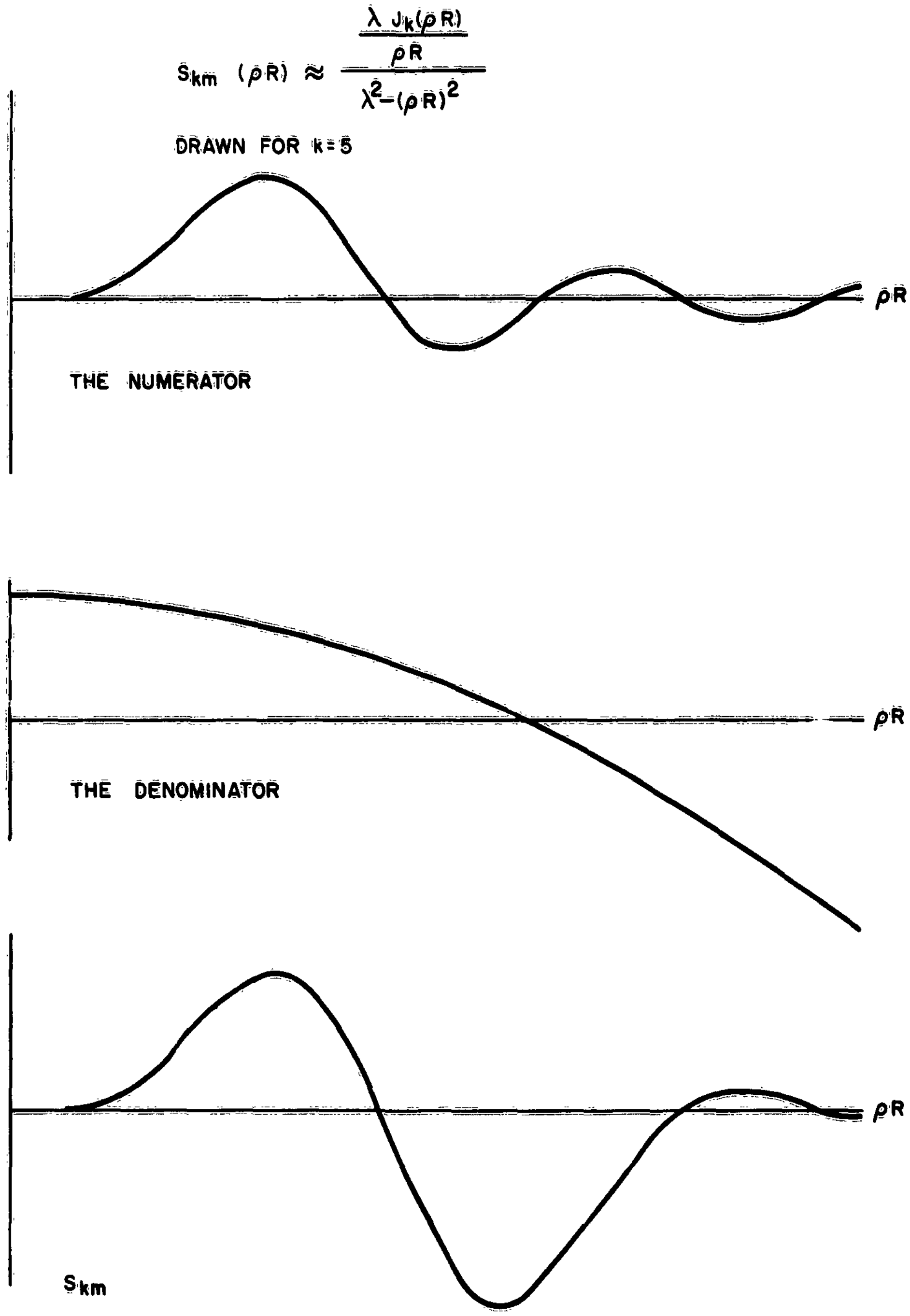

$S_{km}\ (\rho R) \approx \dfrac{\dfrac{\lambda\ J_k(\rho R)}{\rho R}}{\lambda^2 - (\rho R)^2}$
DRAWN FOR k=5
ρR
THE NUMERATOR
ρR
THE DENOMINATOR
ρR
S_{km}

FIG.38

One property of the Radiation functions R_{KM} (ρR), which appears different from the properties of the $\frac{\sin x}{x}$ function is the form of the denominator.

The eigenfunctions for the rectangular case could however be made to have a similar type of denominator if instead of the exponential form of Fourier series a sine and cosine type of series were used. For example, refer to Figure 34.

The Sonine Function (Type III-3)

Consider the function

$$S_{KM}(\rho R) = \Gamma(M+2) \cdot 2^{M+1} \frac{J_{K+M+1}(\rho R)}{(\rho R)^{M+1}}$$

The function S_{KM} (ρR) is a new mathematical function* which will here be called Sonine's function because it is derived from an equation known as Sonine's integral (but in the form shown in Table III). For the case where K=0 the functions have been tabulated and these functions are known as the lambda functions. Curves of the lambda functions taken from Jahnke and Emde are shown in Figure 39. It can be seen that the pattern built up of such a type of function would have a single major lobe and small minor lobes and that size of the minor lobes can be exchanged directly for width of the major lobes.

The general function S_{KM} (ρR) can be demonstrated by two types of sets of curves. The first type is obtained by letting K+M = a constant. In this type there are only a finite number of curves to a set. In fact that number is K+M+2. This type has the useful property that all the functions of each set have the same zeros.

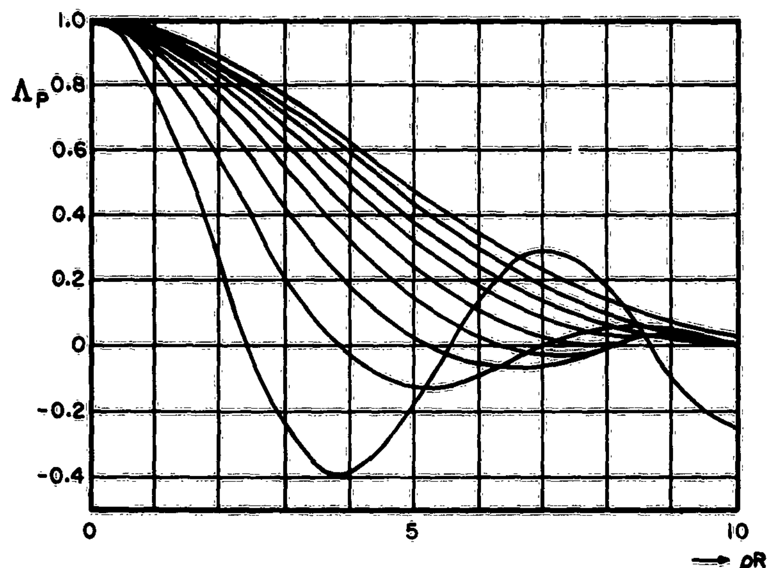
---

*This is $\frac{\Gamma(M+2)}{\Gamma(M+1)}$ = M+1 times the functions which appear in Item 3, Table III. The modification is made for plotting purposes. It can be shown that when M = -1 the Sonine functions give the same pattern as a circular line array, and in fact the illumination for this case is infinite on the perimeter of the circular area, and finite inside. With proper normalization and careful mathematical rigor it can be shown that the circular line array is a limiting case of this sequence.

Note that Γ (M+2) = (M+1)! and that 0! = 1

$$(M+1)\, S_{0,M}(\rho R) = 2^{M+1} \times \underline{M+1|} \times \frac{J_{M+1}(\rho R)}{(\ \ R)^{M+1}} = \Lambda$$

$$\Lambda_P = 2^P\, \underline{P|}\, \frac{J_P(\chi)}{\chi^P}$$



LET $P = M+1$

CURVES SHOWN FOR $M = -1, 0, 1, 2, 3, 4, 5, 6, 7 \longrightarrow$

WHEN $M = -1$, $\Lambda_P = J_P(\chi)$

$M = 0$, $\Lambda_P = \dfrac{2 J_P(\chi)}{\chi}$

FIG. 39

The second type, of which the curves from Jahnke and Emde are an example, have K = a constant. (In that case K = 0.) This group has an infinite number of curves per set, all of which have different zeros, and hence is less useful. Sketches of the six lowest order function sets of the first type are shown in Figure 40. These are drawn here to the same scale as Figure 16 for comparison purposes, although larger curves are used for actual design. It will be noted that each set begins with a Bessel function such as was shown in Figure 16 and ends with a lambda function such as was shown in Figure 39. There is a direct exchange between main lobe shape and side lobe amplitude.

The $S_{KM}(\rho R)$ functions have been used for the synthesis of radiation spectra. Further investigation of these functions leads to other useful properties which, under certain conditions, simplify the synthesis of circular arrays having desired properties. However, such a detailed consideration is beyond the scope of this report.

ARRAYS WITH VERTICAL DIRECTIVITY

Arrays can be designed which have vertical directivity but no azimuthal directivity. The criteria which must be satisfied in order to obtain this type of pattern are as follows:

(1) The arrays shall be circular in shape and the amplitude of the illumination shall be a function only of the radius.

(2) The variation of illumination with angle shall be of the type which has a phase slip of an integral number of cycles going once around the array. This integral shall be independent of radius but it is not necessary that the illumination for all radii shall be in phase.

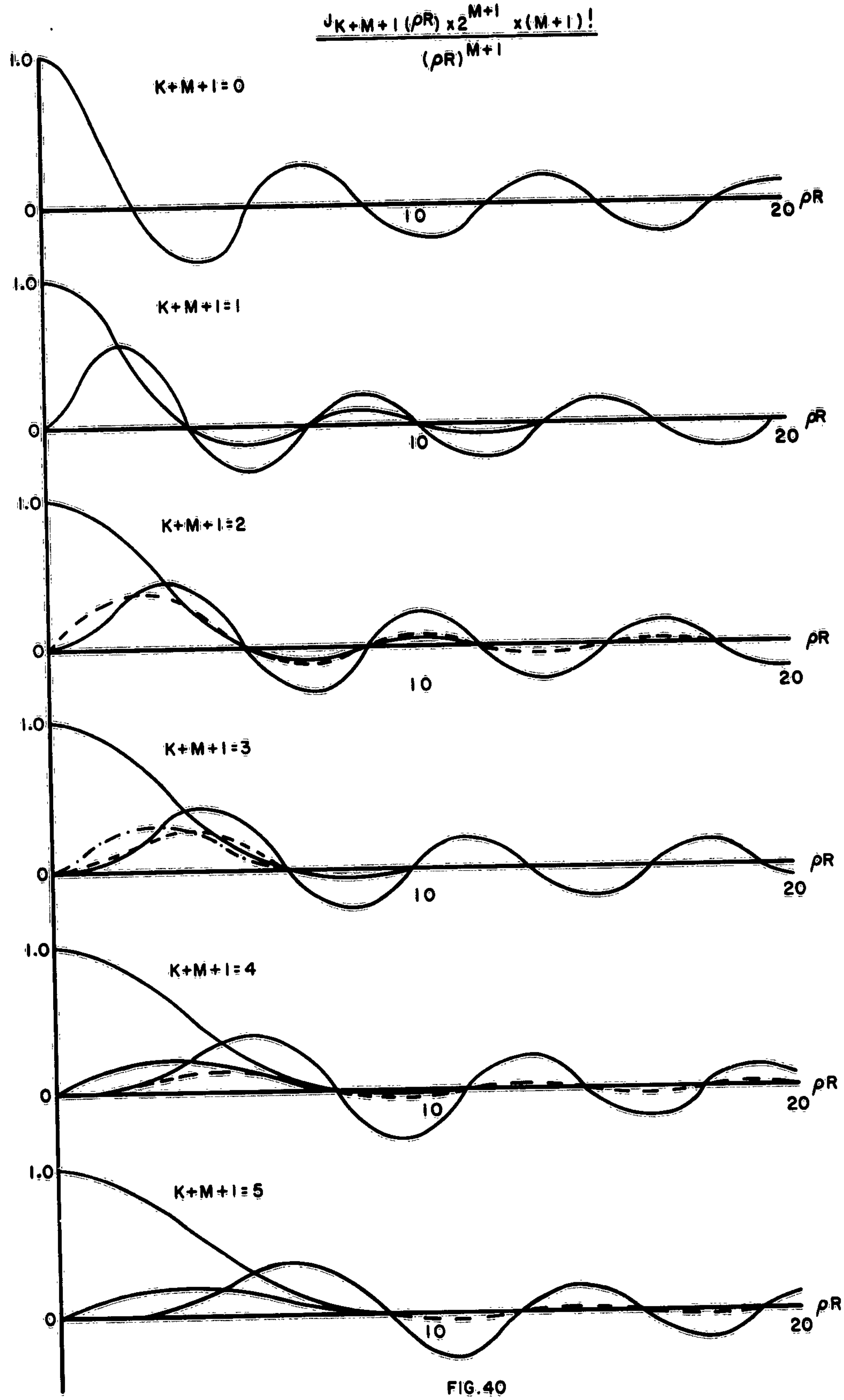

$\frac{J_{K+M+1}(\rho R) \times 2^{M+1} \times (M+1)!}{(\rho R)^{M+1}}$
1.0
0
K+M+1=0
10
20
ρR
K+M+1=1
K+M+1=2
K+M+1=3
K+M+1=4
K+M+1=5

FIG.40

(3) There may be any radial distribution of illumination.

Any one of the circular arrays shown in Tables II, III, and IV may be used. The index number K defines the number of cycles of phase slip around the circle.

THE PLANE AREA REQUIRED FOR A SPECIFIED DIRECTIVITY

The theory which has been developed makes it rather simple to obtain a precise definition of the plane area required in order to synthesize a directive array of a specified beamwidth. This can be done with the aid of a blank design chart of the type shown in Figure 40. By sketching the desired beam shape on Figure 41 and recalling that the radius of a circle is unity at the design frequency the required aperture at the design frequency may be found directly. The relations between beamwidth, aperture, and side lobe level shown in Figure 33 permit a simple estimate which is independent of the exact shape of the array which may eventually be designed for the area.

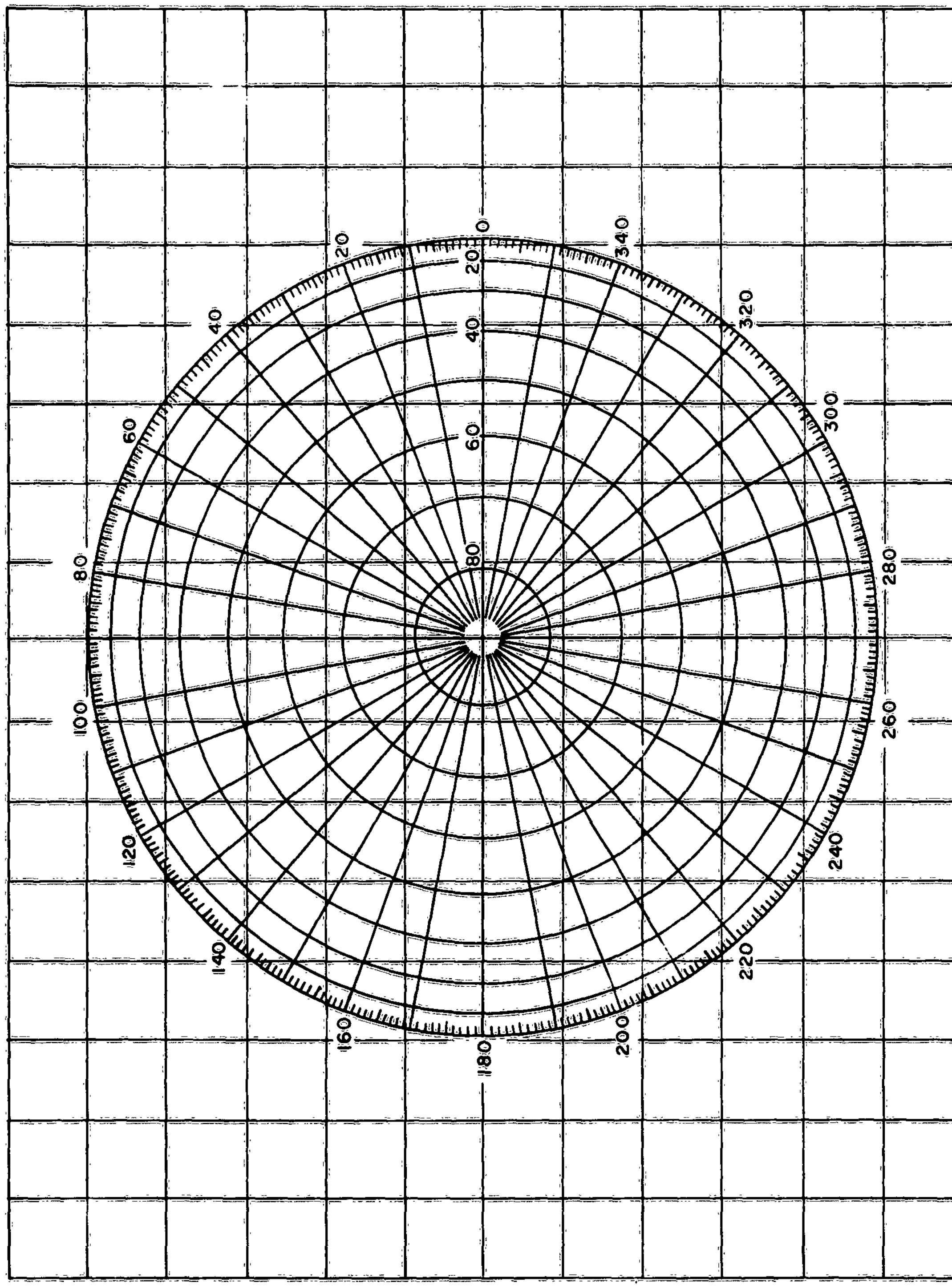

0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
20
40
60
80

FIG. 41

PART IV

CONCLUSION

This report presents a physical and mathematical theory of radiation directivity which permits the synthesis of antenna arrays to produce radiation patterns which may have arbitrarily selected intensities for all directions of radiation. The method is equally powerful as an analysis tool.

The physical limitations of an array are: the region in space which it occupies, the manner in which energy is fed to or taken from it, and the number and type of antennas in the array. Perimeter arrays are not sufficiently general to permit the synthesis of all patterns. Area arrays subject to symmetry restrictions and also arrays of antennas distributed throughout a volume are capable of producing all patterns (in the limit), for all azimuths and elevations.

The region in array space occupied by the antennas is the aperture, and the distribution of current or voltage over the aperture is the illumination. The overall pattern of an array is the product of the pattern per element and the geometrical space factor which results from the distribution of illumination within the aperture.

The space factor for an array is the Fourier Transform of the aperture illumination. The transform of the illumination is representable by a function in the transform plane for the case of a plane array, and as a function in a 3-dimensional transform space in general. Representation of array space by any system of coordinates yields results in transform space in terms of a similar set of coordinates. The basic equation which is used to determine the space factor of an array is obtained by considering the sum of the contributions of all the elements of the

array in the limit as the number of elements becomes infinite, but with the total illumination remaining finite. The integral equations which result are expressible in a set of standard forms depending on the coordinate system and the number of dimensions. When the illumination is expanded in a type of general Fourier series suitable for the particular shape of array, the space factor may be determined in terms of a standard set of functions called eigenfunctions which specifically contain the dimensions of the aperture. The relationships between illuminations and spectrum are tabulated in Tables II, III and IV for several shapes of arrays.

The spectra obtained from rectangular plane arrays are illustrative of the synthesis method. The desired aperture may be synthesized as the sum of a set of eigenfunctions, in terms of which the coordinates which define the aperture illumination are simply determined by the amplitude of the desired space factor at certain points. The spacing between these independent points is inversely proportional to the aperture of the array. The result of using discrete antennas instead of a continuous illumination of the aperture is to make the pattern repeat at regular intervals in transform space. The region in transform space taken up by each of the basic patterns is inversely proportional to the density of the discrete antennas within the aperture. The number of antennas required in an array of arbitrary shape may be readily determined by considering the array to be contained within some convenient rectangular box and thus determining the necessary density of antennas in each coordinate direction to insure freedom from spurious side lobes due to discreteness. Other methods are applicable for special cases such as the circular line array.

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It is necessary in designing arrays to be familiar with standard sets of eigenfunction sequences which may be used to produce desired results. The mathematical forms obtained for circular arrays involve Bessel functions of the first kind and the integrals of some Bessel Functions. However, synthesis of circular arrays is simplified by a study of two special mathematical functions which have the same useful properties as do the $\frac{\sin x}{x}$ functions for rectangular arrays.

With the synthesis method it is possible to determine the necessary illumination for any chosen type of array, and actual synthesis problems must consider the relationships between the shape of the array, the type of basic elements, the orientation of the elements in the array and the effects of reflecting surfaces. For example, dipole patterns may be simply represented in terms of the transform plane for various orientations.

The methods herein presented may be put to advantageous use:

(1) The complete choice of pattern which is available with some types of arrays such as plane area arrays permits the achievement of superior performance to that of some other types of arrays while reducing the cost and complexity.

(2) It is possible to directly exchange aperture for beamwidth, or beamwidth for side lobe level.

(3) Superdirective arrays may be designed in which the intensity in those regions of transform space which do not correspond to radiation space may be arbitrarily varied in order to obtain greater control over the pattern in the region of transform space corresponding to radiation space.

(4) Arrays may be designed which have vertical directivity but no azimuthal directivity.

(5) The plane area required for a specified directivity may be readily found by use of the transform plane.

HAZELTINE ELECTRONICS CORPORATION

Donald Richman

Don Richman, Engineer

APPENDIX A

DERIVATIONS OF EQUATIONS

Table II, Case 1.

Let

$$I(x) = \sum_{K=-\infty}^{\infty} I_K \epsilon^{jK\pi \frac{x}{X}}$$

Then

$$S(u) = \frac{1}{2X} \int_{-X}^{X} I(x) \epsilon^{jux} dx$$

$$= \frac{1}{2X} \sum_{K=-\infty}^{\infty} I_K \left. \frac{\epsilon^{j\left(K\pi \frac{x}{X} + ux\right)}}{jK\frac{\pi}{X} + u} \right|_{-X}^{X}$$

$$= \sum_{K=-\infty}^{\infty} I_K \frac{\sin (uX + K\pi)}{uX + K\pi}$$

Table II, Case 2.

The following general expansion is used here:

$$\epsilon^{j\rho R \cos(\beta-\theta)} = J_0(\rho R) + 2\sum_{K=1}^{\infty} j^{(K)} J_K(\rho R) \cos K(\beta-\theta)$$

$$= J_0(\rho R) + 2\sum_{K=1}^{\infty} (j)^{x} J_K(\rho R) \left[\cos K\beta \cos K\theta + \sin K\beta \sin K\right.$$

Then, since

$$S(\rho,\theta) = \frac{1}{2\pi R}\int_{-\pi}^{\pi} I(\beta)\, \epsilon^{j\rho R \cos(\beta-\theta)}\, d\beta$$

it is convenient to expand $I(\beta)$ into the following Fourier series:

$$I(\beta) = I_0 + I_{11}\cos\beta + I_{12}\cos 2\beta + \cdots + I_{1K}\cos K\beta + \cdots$$
$$+ I_{21}\sin\beta + I_{22}\sin 2\beta + \cdots + I_{2K}\sin K\beta + \cdots$$

This simplifies the integration, since upon multiplying the above two series expansions and integrating over an entire period all of the terms will vanish except those involving terms of the form $\sin^2 K\beta$ or $\cos^2 K\beta$. This leads to the conclusion that,

$$\frac{1}{2\pi R}\int_{-\pi}^{\pi} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} K\beta \cdot \epsilon^{j\rho R\cos(\beta-\theta)}\, d\beta = \frac{1}{R}(j)^K J_K(\rho R) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} K\theta$$

and hence

$$\frac{1}{2\pi R}\int_{-\pi}^{\pi} \epsilon^{jK\beta}\, \epsilon^{j\rho R\cos(\beta-\theta)}\, d\beta = \frac{1}{R}(j)^K J_K(\rho R)\, \epsilon^{jK\theta}$$

For any aperture illumination representable in the form of the Fourier Series, the terms may be integrated separately and the results added.

Table II, Case 3

This is obtained directly from II, 1 by considering the result of 4 linear arrays which are the sides of the rectangle.

Table III, Case 1

This is a direct extension of II, 1 obtained by integrating for each variable in turn.

Table III, Case 2

Let

$$I(r,\phi) = \sum_{K=-\infty}^{\infty} I_K(r)\, \epsilon^{jK\phi}$$

and let

$$I_K(r) = \sum_{M=1}^{\infty} I_{KM}\, J_K\left(\lambda_{KM} \frac{r}{R}\right)$$

where the λ_{KM} are the roots of the equation

$$\lambda_{KM}^{-M}\left[\lambda_{KM} J_K'(\lambda_{KM}) + H J_K(\lambda_{KM})\right] = 0$$

where

$$H = -\frac{1}{R} \left. \frac{\bar{I}_K'(r)}{\bar{I}_K(r)} \right|_{r=R}$$

When H + K is negative an additional term $\mathcal{B}_K(\phi)$ is necessary for this type of

Fourier-Bessel series which is called a Dini Series. Further consideration of this term is beyond the scope of this report. *

Then $$S(\rho,\theta) = \frac{1}{\pi R^2}\int_{-\pi}^{\pi} I(r,\beta)\,\epsilon^{\,j\rho r\cos(\beta-\theta)}\, r\,dr\,d\beta$$

$$= \sum_{K=-\infty}^{\infty} \frac{2}{R^2}(j)^K \epsilon^{jK\theta}\int_0^R J_K(\rho r)\left[\sum_{M=1}^{\infty} I_{KM} J_K\left(\lambda_{KM}\frac{r}{R}\right)\right] r\,dr$$

or since ρ is constant in the integration,

$$S(\rho,\theta) = \sum_{K=-\infty}^{\infty} \frac{2}{\rho R}(j)^K \epsilon^{jK\theta}\int_0^{R\rho} \frac{\rho r}{\rho R} J_K(\rho r)\left[\sum_{M=1}^{\infty} I_{KM} J_K\left(\lambda_{KM}\frac{\rho r}{\rho R}\right) d(\rho r)\right]$$

Let

$$\Psi = \int_0^{\rho R} \frac{\rho r}{\rho R}\, J_K(\rho r)\, J_K\left(\frac{\lambda_{KM}}{\rho R}\rho r\right) d(\rho r)$$

This is a standard form: **

$$\Psi = \frac{\rho r}{\rho R}\left.\frac{J_K\left(\frac{\lambda_{KM}}{\rho R}\rho r\right)\frac{d}{d\rho r}\left[J_K(\rho r)\right] - \frac{\lambda_{KM}}{\rho R} J_K(\rho r)\frac{d}{d\rho r}\left[J_K\frac{\lambda_{KM}}{\rho R}\rho r\right]}{\left(\frac{\lambda_{KM}}{\rho R}\right)^2 - 1}\right|_0^{\rho R}$$

*Bessel Functions, G. N. Watson, Second Edition, MacMillan 1945 Chapter XVIII.

**Bessel Functions, Gray Mathews and MacRoberts, Chapter VI p. 69 eq. 23, 24.

This is zero when $\rho r = 0$, and hence, dropping the subscripts on λ_{KM} for simplicity,

$$\psi = \frac{J_k(\lambda)\frac{d}{d\rho r} J_k(\rho R) - \left(\frac{\lambda}{\rho R}\right)^2 J_k(\rho R)\frac{d}{d\lambda} J_k(\lambda)}{\left(\frac{\lambda}{\rho R}\right)^2 - 1}$$

but

$$\lambda \frac{d}{d\lambda} J_k(\lambda) = -H\, J_k(\lambda)$$

and hence

$$\psi = J_k(\lambda)\,\frac{J_k{}'(\rho R) + \frac{H\lambda}{(\rho R)^2} J_k(\rho R)}{\left(\frac{\lambda}{\rho R}\right)^2 - 1}$$

Since

$$\rho R\, J_k{}'(\rho R) = \rho R\, J_{k-1}(\rho R) - K J_k(\rho R)$$

$$= K J_K(\rho R) - \rho R\, J_{K+1}(\rho R)$$

the following forms for $S(\rho, \theta)$ are obtained by substitution;

$$S(\rho,\theta) = \sum_{K=-\infty}^{\infty} 2(j)^K \epsilon^{jK\theta} \sum_{M=1}^{\infty} I_{KM} J_K(\lambda)\,\frac{\left(H\frac{\lambda}{\rho R} \pm K\right) J_{K+1}(\rho R) \mp \rho R\, J_{K+1}(\rho R}{\lambda^2 - (\rho R)^2}$$

When $\rho R = \lambda$, it is well known that*

$$\int_0^{\rho R} \rho r \, J_K^2(\rho r)\, d(\rho r) = \frac{(\rho R)^2}{2}\left[\left(J_K'(\rho R)\right)^2 + \left(1-\left(\frac{K}{\rho R}\right)^2\right) J_K^2(\rho R)\right]$$

Table III, Case 3

Let
$$S(\rho,\theta) = \sum_{K=-\infty}^{\infty} S_K(\rho,\theta)$$

where
$$S_K(\rho,\theta) = \frac{2}{R^2}(j)^K \epsilon^{jK\theta} \int_0^R r\, I_K(r)\, J_K(\rho r)\, dr$$

It is useful to consider

$$I_K(r) = \sum_{M=0}^{\infty} I_{KM} \left(\frac{r}{R}\right)^K \left(1-\left(\frac{r}{R}\right)^2\right)^M$$

Then let

$$r = R \sin \eta \qquad\qquad dr = R \cos \eta \, d\eta$$

and hence

$$S_K(\rho,\theta) = \frac{2}{R^2}(j)^K \epsilon^{jK\theta} \sum_{M=\infty}^{\infty}\left[\int_0^{\frac{\pi}{2}} I_{KM}\, J_K(\rho R \sin \eta)(\sin \eta)^{K+1}(\cos \eta)^{2M+1}\, d\eta\right]$$

*Bessel Functions, Gray Mathews and Mac Roberts, Chapter VI p. 69 eq. 23, 24.

The term in the brackets is a standard form known as Sonine's First Finite Integral*, and hence

$$S_K(\rho,\theta) = \sum_{K=-\infty}^{\infty} 2(j)^K \epsilon^{jK\theta} \sum_{M=0}^{\infty} I_{KM}\, 2^M\, M!\, \frac{J_{K+M+1}(\rho R)}{(\rho R)^{M+1}}$$

where

$$M! = 1 \cdot 2 \cdot 3 \cdot \cdots (M-1)\, M \cdot \Gamma(M+1)$$

Although this formula is for area arrays and may be used when M = 0, 1, 2 etc., M = -1 gives the proper form of equations for the circular line array (except for a constant multiplier).

Substitution of M = -1 into the equations gives

$$I(r) = \frac{\left(\frac{r}{R}\right)^K}{1-\left(\frac{r}{R}\right)^2}$$

which is finite except when $\frac{r}{R} = 1$. This suggests a modified normalization procedure so that I(r) would define a circular ring for this case.

It may be noted that for this case (-1)! = ∞ . Dividing both I(r) and S(ρ) by -1! indicates that the circular line array, defined by M = -1, may be considered a limiting case of this type of illumination. But this has already been derived.

*Watson, Chapter XII.

Table III, Case 4

Let

$$I(r,\beta)=\sum_{K=-\infty}^{\infty} I_K\left(\frac{r}{R}\right)^{\pm K}\epsilon^{jK\beta}$$

then

$$S(\rho,\theta)=\frac{1}{\pi R_o^2}\int_{-\pi}^{\pi}\int_{R_1}^{R_o} I(r,\beta)\epsilon^{j\rho r\cos(\beta-\theta)}\, r\,dr\,d\theta$$

$$=\frac{2}{R_o}\sum_{K=-\infty}^{\infty} I_K\,(j)^K\epsilon^{jK\theta}\int_{R_1}^{R_o}\left(\frac{r}{R_o}\right)^{\pm K+1} J_K(\rho r)\,dr$$

$$=\frac{2}{\rho R_o}\sum_{K=-\infty}^{\infty} I_K(j)^K\epsilon^{jK\theta}\int_{R_1}^{R_o}\left(\frac{\rho r}{\rho R_o}\right)^{\pm K+1} J_K(\rho r)\,d(\rho r)$$

This is a standard form*: (and hence)

$$S(\rho,\theta)=2\sum_{K=-\infty}^{\infty} I_K\,(j)^K\epsilon^{jK\theta}\left[\left(\frac{\rho r}{\rho R_o}\right)^{\pm K+1}\frac{J_{\pm K+1}(\rho r)}{\rho R_o}\right]\Bigg|_{R_1}^{R_o}$$

This is useful for cases in which negative powers of $\left(\frac{r}{R_o}\right)$ require that the illumination be defined as different from zero only in rings which enclose the origin.

Table III, Case 5

This is an extension of III, 1 and II, 3.

Table III, Case 6

This is a combination of II, 1 and II, 2.

*Tables of Integral and Other Mathematical Data, H. B. Dwight, Revised Edition, Macmillan, 1947. #835.1

Table IV, Case 1

This is a direct extension of II, 1 and III, 1.

Table IV, Case 2

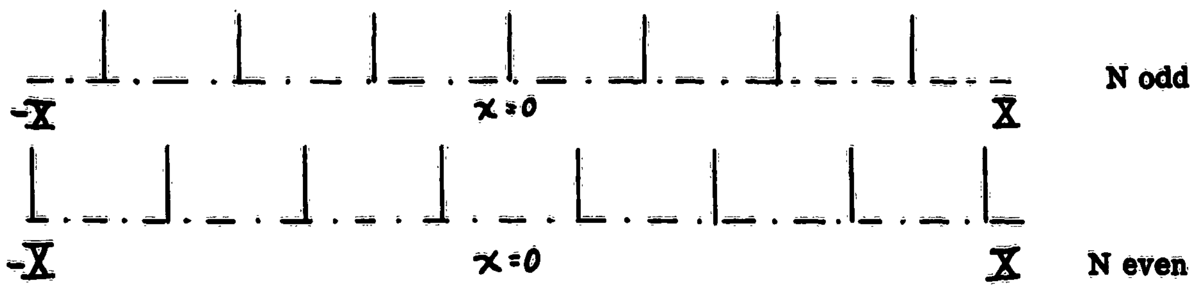
This is a combination of II, 1 and III, 4, III, 5, or III, 6.

APPENDIX B

DERIVATIONS OF FORMULAS FOR ARRAYS OF DISCRETE ANTENNAS

(1) Linear arrays.

The pulse function P (x) is shown below: N is the number of antennas.



When N is odd

$$P(x) = \sum_{p=-\infty}^{\infty} \epsilon^{jNp\pi \frac{x}{X}}$$

When N is even

$$P(x) = \sum_{p=-\infty}^{\infty} \epsilon^{j\left(Np\pi \frac{x}{X} + p\pi\right)}$$

Since

$$(-1)^{(N+1)p} = \begin{cases} \epsilon^{jp\pi} & N \text{ even} \\ 1 & N \text{ odd} \end{cases}$$

then, whether N is even or odd,

$$P(x) = \sum_{p=-\infty}^{\infty} (-1)^{(N+1)p} \epsilon^{jNp\pi \frac{x}{X}}$$

The function $I_0(x)$ is given by

$$I_0(x) = \sum_{K=-\infty}^{\infty} I_K \epsilon^{jK\pi \frac{x}{X}}$$

and since $I(x) = I_0(x)\ \rho(x)$

$$I(x) = \sum_{K=-\infty}^{\infty} \sum_{\rho=-\infty}^{\infty} (-1)^{(N+1)\rho}\, I_K\, \epsilon^{j(N\rho+K)\pi\frac{x}{X}}$$

and hence

$$S(u) = \sum_{K=-\infty}^{\infty} \sum_{\rho=-\infty}^{\infty} (-1)^{(N+1)\rho}\, I_K\, \frac{\sin(uX+K\pi+N\rho\pi)}{uX+K\pi+N\rho\pi}$$

For the Kth harmonic component of the aperture illumination, the spectial component for the case of a discrete array is

$$S_K(u) = I_K \sum_{\rho=-\infty}^{\infty} (-1)^{(N+1)\rho}\, \frac{\sin(uX+K\pi+N\rho\pi)}{uX+K\pi+N\rho\pi}$$

The effect of discreteness is to make the pattern repeat at intervals of N independent points. The effect of discreteness depends on the number of antennas only in degree, hence, because the side lobes of the $\frac{\sin uX}{uX}$ function measured at any point alternate in sign depending on whether N is even or odd, the $(-1)^{(N+1)\rho}$ term occurs in the equation to make the phase of the term added by discreteness independent, at any point, of the number of antennas; for example see Figure 14.

The formulas may also be written in closed form. For example, the well known result for a uniformly illuminated linear array of discrete antennas would be written, in the symbols used here as*

$$\frac{\sin uX}{\sin \frac{uX}{N}}$$

*See "Electromagnetic Waves", Schelkunoff, Van Nostrand, p. 342, 353.

This could be used as an eigenfunction but it has the disadvantage that it is a function of N. It is more convenient to use the "universal" $\frac{\sin uX}{uX}$ functions for synthesis, although the above may be used as a computing aid.

(2) Rectangular Arrays.

The pulse function is

$$P(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=\infty}^{\infty}(-1)^{[(N_x+1)p+(N_y+1)q]}\,\epsilon^{j(N_x p\pi\frac{x}{X}+N_y q\pi\frac{y}{Y})}$$

and since

$$I_0(x,y)=\sum_{K=-\infty}^{\infty}\sum_{M=-\infty}^{\infty}I_{KM}\,\epsilon^{j(K\pi\frac{x}{X}+M\pi\frac{y}{X})}$$

Then

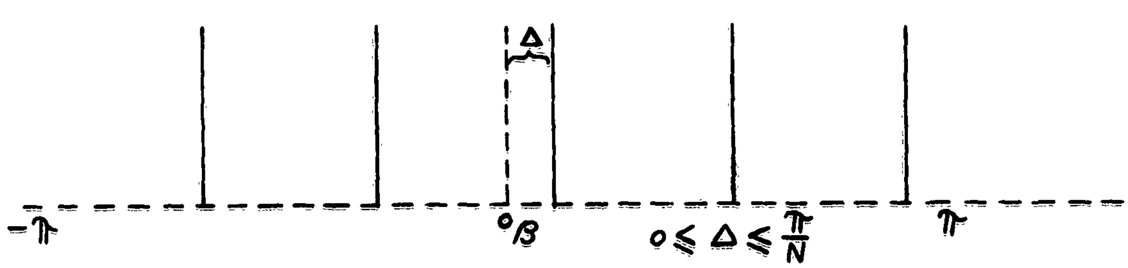
$$I(x,y)=\sum_K\sum_M^{\infty}\sum_p\sum_q I_{KM}(-1)^{(Nx+1)p+(Ny+1)q}\,\epsilon^{j\pi[(Nxp+K)\frac{x}{X}+(Nyq+M)\frac{y}{Y}]}$$

$$S(u,v)=\sum_K\sum_{M\,-\infty}^{\infty}\sum_p\sum_q I_{KM}(-1)^{(Nx+1)p+(Ny+1)q}\frac{\sin(uX+K\pi+N+p\pi)\,\sin(vY+M\pi+Nqq\pi)}{(uX+K\pi+Nxp\pi)(uY+M\pi+Nyq\pi)}$$

This is a two dimensional extension of the preceding case. See Figure 11.

(3) The Circular Line Array.

P (ρ) for this case is found from the following diagram:



When Δ is zero;

$$P(\beta) = \sum_{\rho=-\infty}^{\infty} \epsilon^{jN\rho\beta}$$

in general;

$$P(\beta) = \sum_{\rho=-\infty}^{\infty} \epsilon^{jN\rho(\beta+\Delta)}$$

then

$$I(\beta) = \sum_{K=-\infty}^{\infty} \sum_{\rho=-\infty}^{\infty} \epsilon^{j(K+\rho N)\beta+\rho N\Delta}$$

where Δ is a constant.

Then the following formula from Table 1 is used. ($r = R$ here)

$$S(\rho,\theta) = \frac{1}{2\pi R} \int_{-\pi}^{\pi} I(\beta)\, \epsilon^{j\rho R\cos(\beta-\theta)}\, d\beta$$

In order to perform the integration, the following two formulae are required.

(1)

$$\epsilon^{j\rho R\cos(\beta-\theta)} = J_0(\rho R) + 2\sum_{q=1}^{\infty} (j)^q J_q(\rho R)\cos q(\beta-\theta)$$
$$= J_0(\rho R) + 2\sum_{q=1}^{\infty} (j)^q J_q(\rho R)\,[\cos q\beta \cos q\theta + \sin q\beta \sin$$

This is a trigonometric Fourier series in β, with Bessel function coefficients which are constant for the integration.

$$\epsilon^{j(K+\rho N)\beta+\rho N\Delta} = \epsilon^{j\rho N\Delta}(\cos(K+\rho N)\beta + j\sin(K+\rho N)\beta)$$

When the two series are multiplied together and integrated, only the terms in $\cos^2(K+\rho N)\beta$ and $\sin^2(K+\rho N)\beta$ will give results different from zero.

Then

$$S(\rho,\theta) = \frac{1}{R} \sum_{K=-\infty}^{\infty} \sum_{\rho=-\infty}^{\infty} \epsilon^{j\rho N\Delta} (j)^{\rho N+K} \left(J_{\rho N+K}(\rho R)\right) \epsilon^{j(\rho N+K)\theta}$$

For the Kth harmonic

$$S_K(\rho,\theta) = \frac{1}{R} \sum_{\rho=-\infty}^{\infty} (j)^{(\rho N+K)} \epsilon^{j\rho N+K} J_{\rho N+K}(\rho R) \epsilon^{j(\rho N+K)\theta}$$

The meaning of this last form is made clearer if the terms are written out for three specific cases of the illumination. Let $\Delta = 0$ for simplicity. (When Δ is not zero the "correction" pattern due to discreteness is rotated by the angle Δ.)

Case 1

Let $I(\beta) = R =$ Constant (Uniform illumination)

then

$$\begin{aligned} S(\rho,\theta) = J_0(\rho R) & \\ & + 2(j)^N J_N(\rho R)\cos N\theta \\ & + 2(j)^{2N} J_{2N}(\rho R)\cos 2N\theta \\ & + 2(j)^{3N} \ldots\ldots \end{aligned}$$

When the antennas are spaced 1/2 wave length apart around the circle, the highest value of ρR which normally might correspond to real radiation space is $\rho R = N$. Reference to Figure 16 shows why the terms of $J_{2N}(\rho R)$ and higher orders can be neglected.

For example,

$$J_{36}(18) = 0.000000006335$$

This effect is not quite as drastic for the lower orders, for example

$$J_{10}(10) = 0.2075$$
$$J_{20}(10) = 0.00001151$$

and $$J_{30}(10) = 0.000000000001551$$

Thus the series is rapidly convergent in the important region near the origin, where ρR is small.

Case 2

Let $I(\beta) = R \cos K\beta$

then

$$\begin{aligned} S(\rho,\theta) &= (j)^K J_K(\rho R) \cos K\theta \\ &+ (j)^{N+K} J_{N+K}(\rho R) \cos(N+K)\theta \\ &+ (j)^{N-K} J_{N-K}(\rho R) \cos(N-K)\theta \\ &+ \cdots \end{aligned}$$

Case 3

Let $I(\beta) = R \sin K\beta$

then

$$\begin{aligned} S(\rho,\theta) &= (j)^K J_K(\rho R) \cos K\theta \\ &+ (j)^{N+K} J_{N+K}(\rho R) \sin(N+K)\theta \\ &- (j)^{N+K} J_{N-K}(\rho R) \sin(N-K)\theta \\ &+ \cdots \end{aligned}$$

In all cases the first term is what would be obtained with a continuously illuminated aperture, and only the first pair of correction terms need usually be collected.

APPENDIX C

TABLE OF SYMBOLS

(1) COMPARATIVE TABLE

Two groups of symbols are shown below. One group tabulates the symbols which represent dimensions and functions in array space. The other group is related to transform space. Pairs of functions which appear opposite each other in the table are analogous, or have similar significance in the two spaces. The capital letters, X, Y, Z, R represent fixed, usually maximum values of the variables.

Array Space		Transform Space
x, X	Linear Dimension	u
y, Y	Linear Dimension	v
z, Z	Linear Dimension	w
r, R	Radius from Origin	ρ
β	Angle, counterclockwise from the $\begin{Bmatrix} x \\ u \end{Bmatrix}$ axis	θ
$I(x, y, z)$ or $I(r, \beta, z)$ The illumination	Fourier Transform of	The Spectrum $S(u, v, w)$ or $S(\rho, \theta, w)$
$\left.\begin{matrix} \alpha \\ \\ h \end{matrix}\right\}$ Directions in array radiation space	Azimuth angle elevation angle	Directions from center of radiation sphere to point on surface defining in array radiation space $\left\{\begin{matrix} \alpha \\ \\ h \end{matrix}\right.$

(2) ALPHABETICAL TABLE

Symbol	Description
$B_k(\beta)$	Initial term of Dini Series
f	Radio frequency
f_o	Design-center radio frequency
$F(\alpha, h)$	Radiation pattern
$F_o(\alpha, h)$	The radiation pattern of a single antenna
$F(f)$	Campbell-Foster form of Fourier Transform
$G(g)$	Campbell-Foster form of Fourier Transform
h	Elevation angle
h_o	Elevation angle of nominal beam direction
H	A parameter defined on page
I (region)	The illumination in a specified region
$I(x)$	Aperture illumination for a linear array along the x axis
$\lvert I(x) \rvert$	Amplitude of $I(x)$
$I(x, y)$	Aperture illumination of a rectangular array in the (x, y) plane
$I(x, y, z)$	Aperture illumination for a rectangular volume array with (x, y, z) coordinates
$I(\beta)$	Aperture illumination of a circular line array
$I(r, \beta)$	Aperture illumination of a circular plane array
$I(r, \beta, z)$	Aperture illumination of a cylindrical array
$I_k(\beta)$	Fourier Bessel form of aperture illumination
$I_k(r)$	Radial function of the Kth harmonic component of aperture illumination
I_{KM}	Fourier coefficient
$I_o(x)$	Envelope of aperture illumination of a discrete linear array

(2) ALPHABETICAL TABLE (Cont'd)

Symbol	Description
$I_o(x, y)$	Envelope of aperture illumination for a rectangular array of discrete antennas
$J_o(\rho R)$	Bessel function of first kind of order zero and argument ρR
$J_{10}(\rho R)$	Bessel function of first kind of order ten and argument ρR
$J_K(K)$	Bessel function of first kind with argument equal to the order
K	An integer, an index number
L	An integer, an index number
M	An integer, an index number
N	An integer
N_x	Repetition rate of patterns, equal to the number of rows along x axis
N_y	Repetition rate of patterns, equal to the number of rows along y
ρ	An integer, an index number
p-plane	The complex plane of contour integration
P(x)	Aperture illumination impulse function for discrete antennas along x axis
P(x, y)	Aperture illumination impulse function for discrete antennas in x, y plane
q	An integer, an index number
r	Radial coordinate in array space
R	Outer or constant radius of a circle
R_o	Outside radius of an annular ring
R_1	Inside radius of an annular ring

(2) ALPHABETICAL TABLE (Cont'd)

Symbol	Description
$R_{KM}(\rho R)$	Radiation function
$S(u)$	Space factor in terms of u
$S(u, v)$	Space factor in terms of u, v
$S(\rho, \theta)$	Space factor in terms of ρ, θ
$S(\alpha, h)$	Space factor in terms of α, h
$S_K(\rho, \theta)$	Kth harmonic component of space factor
$S_{KM}(u)$	Kth harmonic component of space factor
$S_{KM}(u, v)$	Kth harmonic component of space factor
$S_{KM}(\rho R)$	Sonine function
t	time
t_o	reference time
u	coordinate in transform space defined on page
u'	coordinate in transform space defined on page
v	coordinate in transform space defined on page
w	coordinate in transform space defined on page
x	coordinate in array space
X	Semi-aperture in the x direction
y	coordinate in array space
Y	Semi-aperature in the y direction

(2) ALPHABETICAL TABLE (Cont'd)

Symbol	Description
z	Coordinate in array space
Z	Semi-aperture in the Z direction
α	Azimuth angle
α_o	Azimuth angle of nominal beam direction
β	Angle in array space
$\Gamma(M+2)$	The gamma function of argument (M + 2)
η	Variable of integration
θ	Angle in the transform plane
λ_{KM}	A parameter
ρ	Radial coordinate in the transform plane
$\phi(x)$	A phase function in the aperture illumination of a linear array
$\Delta\phi$	A phase difference
$\phi_K'(p)$	A Bessel transform
ϕ (region)	Phase function defines in a region
ψ	Integral defined on page
	$2\pi f$